# A FAMILY OF KOSZUL ALGEBRAS ARISING FROM FINITE-DIMENSIONAL REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra and let $\mathbf{S}^{\mathfrak{g}}$ be the locally finite part of the algebra of invariants $(\operatorname{End}_{\mathbf{C}} \mathbf{V} \otimes S(\mathfrak{g}))^{\mathfrak{g}}$ where $\mathbf{V}$ is the direct sum of all simple finite-dimensional modules for $\mathfrak{g}$ and $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. Given an integral weight $\xi$, let $\Psi = \Psi(\xi)$ be the subset of roots which have maximal scalar product with $\xi$. Given a dominant integral weight $\lambda$ and $\xi$ such that $\Psi$ is a subset of the positive roots we construct a finite-dimensional subalgebra $\mathbf{S}^{\mathfrak{g}}_{\Psi}(\leq_{\Psi} \lambda)$ of $\mathbf{S}^{\mathfrak{g}}$ and prove that the algebra is Koszul of global dimension at most the cardinality of $\Psi$. Using this we then construct naturally an infinite-dimensional Koszul algebra of global dimension equal to the cardinality of $\Psi$. The results and the methods are motivated by the study of the category of finite-dimensional representations of the affine and quantum affine algebras.

## Introduction

This paper is motivated by the study of the category of finite-dimensional representations of both the classical and quantum loop algebras associated to a simple Lie algebra $\mathfrak{g}$. This category is not semisimple and thus it is natural to investigate its homological properties. However, this category is both too large in the sense that it has too many simple objects and too small in the sense that it does not have enough projectives. This means that one of the classical tools of representation theory, namely replacing a category by the category of modules over the endomorphism ring of its projective generator, is not available to us. This tool plays a very important role in the study of the category $\mathcal{O}$ (cf. [1] and more recently [2, 3, 11, 16, 17] to name but a few) and in many other situations as well. It is also well-known that the endomorphism ring of a projective generator is often a nice associative algebra. For instance, in the case of category $\mathcal{O}$ that algebra is Koszul. Thus another motivation for this paper was to find an appropriate category of finite-dimensional modules which would allow us to use these methods.

There are many important families of irreducible representations of the quantum affine algebra, such as the Kirillov-Reshetikhin modules ([13]) or more generally the minimal affinizations associated to dominant integral weights in the weight lattice of the simple Lie algebra ([4]), which on specializing to $q = 1$ become indecomposable modules for a truncated loop algebra ([5]). In most cases, they are in fact modules for $\mathfrak{g} \otimes \mathbf{C}[t^{\pm 1}]/((t-1)^2)$ and admit a natural $\mathbf{Z}_+$-grading given by powers of $(t-1)$. In other words, some important families of modules can be regarded as $\mathbf{Z}_+$-graded modules for the finite-dimensional $\mathbf{Z}_+$-algebra $\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ad}}$, where

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$\mathfrak{g}_{\rm ad}$ has degree 1 and is an abelian ideal. It is therefore reasonable to restrict our attention to the subcategory $\mathcal{G}_2$ of graded modules with finite-dimensional graded pieces over this quotient of the loop algebra. This category still has infinitely many simple objects but they are now parametrized discretely and admit projective covers. In fact, we show that every simple module has an explicit projective resolution, coming from the Koszul complex of the symmetric algebra of $\mathfrak{g}$, and this allows us to compute all Ext spaces.

To construct interesting finite-dimensional associative algebras we have to pass to Serre subcategories which contain only finitely many simple objects. Using the description of the Kirillov-Reshetikhin modules in terms of generators and relations given in [5, 8], one can naturally associate to such a module a subset $\Psi_i$ of $R^+$. This subset maximizes the linear functional on the dual of a Cartan subalgebra of $\mathfrak{g}$ given by the scalar product with the $i$th fundamental weight $\omega_i$. It is then natural to consider the case when $\omega_i$ is replaced by an arbitrary integral weight $\xi$. We denote the corresponding set by $\Psi(\xi)$. These sets have many interesting combinatorial properties which are studied in [6]. For instance, in some cases the maximal possible cardinality of such a set equals the maximal dimension of a nilpotent abelian subalgebra of $\mathfrak{g}$ which were computed already in [14].

Suppose now that $\xi$ is such that $\Psi = \Psi(\xi)$ is contained in a fixed set of positive roots of $\mathfrak{g}$. This is always the case if $\xi$ is dominant and in fact in this case $\Psi$ defines an abelian ideal in the corresponding Borel subalgebra. We define a partial order $\leq_\Psi$ on the set of dominant integral weights which is a refinement of the usual partial order. Let $\mathcal{G}_2[\leq_\Psi \lambda]$ be the subcategory of $\mathcal{G}_2$ consisting of objects whose irreducible constituents are in $\leq_\Psi$. We show that $\mathcal{G}_2[\leq_\Psi \lambda]$ has enough projectives and is of global dimension at most equal to $|\Psi|$ and the bound is attained if $\lambda$ is sufficiently dominant. The endomorphism ring of a projective generator of $\mathcal{G}_2[\leq_\Psi \lambda]$ admits a natural grading and we are able to prove by using the results of [2] that this grading is Koszul. We are also able to identify the endomorphism algebra as a subalgebra of $\mathfrak{g}$-invariants in the tensor product algebra $\operatorname{End}_{\mathbf{C}} \mathbf{V} \otimes S(\mathfrak{g})$ where $\mathbf{V}$ is the direct sum of all simple finite-dimensional modules for $\mathfrak{g}$. Finally we prove that these algebras "approximate" an infinite-dimensional algebra which is also Koszul and describe the the corresponding Yoneda algebras explicitly. These algebras can be realized as path algebras of rather simple quivers with relations (cf. example in 1.7 and [10]).

The paper is organized as follows. In Section 1 we formulate the main results of the paper and briefly explain the strategy for proving them. In Section 2 we study the fundamental properties of the category $\mathcal{G}_2$ and a family of its Serre subcategories. In Section 3 we proceed to investigate the homological properties of the category $\mathcal{G}_2$. The next two sections are dedicated to studying the endomorphism algebras and their quadratic duals.

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## 1. Main results

Throughout this paper $\mathbf{Z}_+$ stands for the set of non-negative integers and $\mathbf{C}$ denotes the field of complex numbers. All algebras and vector spaces, as well as Hom and tensor products,

are considered over $\mathbf{C}$ unless specified otherwise. If $A$ is an associative algebra, we denote $A^{op}$ its opposite algebra.

**1.1.** Let $\mathfrak{g}$ be a complex finite dimensional simple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and let $R \subset P \subset \mathfrak{h}^*$ be, respectively, the set of roots and the weight lattice of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $(\cdot,\cdot)$ be the non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ induced by the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$. Set $I = \{1,\dots,\dim\mathfrak{h}\}$ and let $\alpha_i$, $i \in I$ be a set of simple roots and $\omega_i$, $i \in I$ the corresponding fundamental weights. Let $R^+$ be the corresponding set of positive roots and let $P^+$ be the $\mathbf{Z}_+$-span of the fundamental weights.

**1.2.** For $\lambda \in P^+$, let $V(\lambda)$ be the unique, up to isomorphism, simple finite dimensional $\mathfrak{g}$-module with highest weight $\lambda$ and let $V(\lambda)^* = \operatorname{Hom}(V(\lambda),\mathbf{C})$ with its standard $\mathfrak{g}$-module structure. Set

$$\mathbf{V} = \bigoplus_{\lambda\in P^+} V(\lambda), \qquad \mathbf{V}^{\circledast} = \bigoplus_{\lambda\in P^+} V(\lambda)^*.$$

Note that $\mathbf{V}^{\circledast} \cong \mathbf{V}$ as a $\mathfrak{g}$-module. Define an associative algebra structure on $\mathbf{V}^{\circledast}\otimes\mathbf{V}$ by extending linearly

$$(f\otimes v)(g\otimes w) = g(v) f\otimes w, \qquad v,w\in\mathbf{V},\ f,g\in\mathbf{V}^{\circledast}.$$

The natural embedding $\mathbf{V}^{\circledast}\otimes\mathbf{V}\to\operatorname{End}\mathbf{V}$ (respectively, $\mathbf{V}^{\circledast}\otimes\mathbf{V}\to\operatorname{End}\mathbf{V}^{\circledast}$) of $\mathfrak{g}$-modules given by extending $f\otimes v\mapsto (w\mapsto f(w)v)$ (respectively, $f\otimes v\mapsto (g\mapsto g(v)f)$) is then an anti-homomorphism of associative algebras. In particular, for all $\lambda\in P^+$ we have isomorphisms of associative algebras

$$V(\lambda)^*\otimes V(\lambda)\to(\operatorname{End}V(\lambda))^{op}, \qquad V(\lambda)^*\otimes V(\lambda)\to(\operatorname{End}V(\lambda)^*)^{op}.$$

The preimage of the identity element in $\operatorname{End}V(\lambda)^{op}$ (or $(\operatorname{End}V(\lambda)^*)^{op}$) is the canonical $\mathfrak{g}$-invariant element $1_\lambda$ of $V(\lambda)^*\otimes V(\lambda)$.

**Lemma.** *For $\lambda,\mu\in P^+$ we have*

$$1_\lambda 1_\lambda = 1_\lambda, \qquad 1_\lambda 1_\mu = 0, \quad \mu\neq\lambda$$

*and*

$$1_\lambda(\mathbf{V}^{\circledast}\otimes\mathbf{V}) = V(\lambda)^*\otimes\mathbf{V}, \qquad (\mathbf{V}^{\circledast}\otimes\mathbf{V})1_\mu = \mathbf{V}^{\circledast}\otimes V(\mu).$$

*Proof.* Observe first that $(f\otimes v)(g\otimes w)=0$ if $v\in V(\nu)$, $g\in V(\xi)^*$ with $\xi\neq\nu$. Write $1_\lambda = \sum_i \xi_i\otimes u_i$, $\xi_i\in V(\lambda)^*$, $u_i\in V(\lambda)$. Then we have $\sum_i\xi_i(v)u_i = v$ for all $u\in V(\lambda)$ and $\sum_i \xi_i f(u_i) = f$ for all $f\in V(\lambda)^*$. It follows that $1_\lambda(f\otimes v) = f\otimes v$ for all $f\in V(\lambda)^*$, $v\in\mathbf{V}$ and that $(g\otimes u)1_\lambda = g\otimes u$ for all $g\in\mathbf{V}^{\circledast}$, $u\in V(\lambda)$. Since $\mathbf{V}^{\circledast}\otimes\mathbf{V} = \bigoplus_{\nu,\xi\in P^+} V(\nu)^*\otimes V(\xi)$, the assertions follow. $\square$

**1.3.** Let $A$ be any associative algebra with unity $1_A$ and consider $\mathbf{A} = A\otimes(\mathbf{V}^{\circledast}\otimes\mathbf{V})$. This is obviously an associative algebra. If $A$ is a $\mathfrak{g}$-module algebra, that is the multiplication map $A\otimes A\to A$ is a homomorphism of $\mathfrak{g}$-modules, then the usual tensor product action defines a $\mathfrak{g}$-module structure on $\mathbf{A}$ and the algebra multiplication is again a morphism of $\mathfrak{g}$-modules. By abuse of notation we let $1_\lambda$ also denote the idempotent $1_A\otimes 1_\lambda$ in $\mathbf{A}$. For $\lambda,\mu\in P^+$ we have

$$1_\lambda\mathbf{A}1_\mu = A\otimes V(\lambda)^*\otimes V(\mu).$$

If the algebra $A$ has a $\mathbf{Z}_+$-grading $A = \bigoplus_{k\in\mathbf{Z}_+} A[k]$ which is compatible with the $\mathfrak{g}$-action, then $\mathbf{A}$ has a natural grading given by

$$\mathbf{A}[k] = A[k] \otimes (\mathbf{V}^{\circledast} \otimes \mathbf{V}).$$

*From now on we shall assume that $A$ is a $\mathbf{Z}_+$-graded associative $\mathfrak{g}$-module algebra with unity and also that $A[0] = \mathbf{C}1_A$, $\dim A[k] < \infty$, $k \in \mathbf{Z}_+$.*

**1.4.** Given $\Psi \subset R^+$, let $\le_\Psi$ be the partial order on $P^+$ given by:

$$\lambda \le_\Psi \mu \iff \mu - \lambda \in \mathbf{Z}_+\Psi,$$

where $\mathbf{Z}_+\Psi$ is the non-negative integer linear span of the elements of $\Psi$. This is a refinement of the usual partial order $\le = \le_{R^+}$ on $P^+$. Given $\lambda, \mu \in P^+$, set

$$\le_\Psi \lambda = \{\nu \in P^+ : \nu \le_\Psi \lambda\}, \qquad \lambda \le_\Psi = \{\nu \in P^+ : \lambda \le_\Psi \nu\},$$
$$[\mu, \lambda]_\Psi = (\le_\Psi \lambda) \cap (\mu \le_\Psi).$$

The first set and hence the third are finite subsets of $P^+$.

For all $\lambda, \mu \in P^+$ with $\lambda \le_\Psi \mu$, define

$$d_\Psi(\lambda, \mu) = \min\{\sum_{\beta\in\Psi} m_\beta : \mu - \lambda = \sum_{\beta\in\Psi} m_\beta \beta,\ m_\beta \in \mathbf{Z}_+\}.$$

Given $\lambda, \mu \in P^+$ with $\lambda \le_\Psi \mu$, set

$$\mathbf{A}_\Psi(\lambda, \mu) = 1_\lambda \mathbf{A}[d_\Psi(\lambda, \mu)] 1_\mu$$

and for $F \subset P^+$, set

$$\mathbf{A}_\Psi(F) = \bigoplus_{\lambda,\mu\in F\,:\,\lambda\le_\Psi\mu} \mathbf{A}_\Psi(\lambda, \mu).$$

Clearly $\mathbf{A}_\Psi(\lambda, \mu)$ and $\mathbf{A}_\Psi(F)$ are $\mathfrak{g}$-submodules of $\mathbf{A}$. The following Lemma is easily checked.

**Lemma.** *Suppose that $\Psi \subset R^+$ is such that for all $\lambda, \mu, \nu \in P^+$ with $\lambda \le_\Psi \mu \le_\Psi \nu$, we have*

$$d_\Psi(\lambda, \mu) + d_\Psi(\mu, \nu) = d_\Psi(\lambda, \nu).$$

*Then for all $F \subset P^+$ the $\mathfrak{g}$-submodule $\mathbf{A}_\Psi(F)$ is a graded subalgebra of $\mathbf{A}$.* $\square$

It is easy to construct examples of sets $\Psi$ satisfying the conditions of the Lemma. Thus for $\xi \in P$, let

$$\max \xi = \max\{(\alpha, \xi) : \alpha \in R\}, \qquad \Psi(\xi) = \{\alpha \in R : (\xi, \alpha) = \max \xi\}.$$

Clearly if $\xi \ne 0$, we have

$$\max \xi > 0, \qquad (\Psi(\xi) + \Psi(\xi)) \cap (R \cup \{0\}) = \emptyset.$$

Let $\Psi = \Psi(\xi)$ and assume that $\Psi \subset R^+$ (note that this holds if $\xi \in P^+$). Then we see that for all $\lambda, \mu, \nu \in P^+$ with $\lambda \le_\Psi \mu \le_\Psi \nu$ we have

$$d_\Psi(\lambda, \mu) + d_\Psi(\mu, \nu) = d_\Psi(\lambda, \nu).$$

**1.5.** Let $\mathbf{A}^{\mathfrak{g}}$ be the submodule of $\mathfrak{g}$-invariants of $\mathbf{A}$. For $\lambda,\mu\in P^+$, $F\subset P^+$, and $\Psi\subset R^+$ set

$$\mathbf{A}^{\mathfrak{g}}_{\Psi}(\lambda,\mu)=(\mathbf{A}_{\Psi}(\lambda,\mu))^{\mathfrak{g}},\qquad \mathbf{A}^{\mathfrak{g}}_{\Psi}(F)=(\mathbf{A}_{\Psi}(F))^{\mathfrak{g}}.$$

The following is clear.

**Proposition.** (i) *The submodule* $\mathbf{A}^{\mathfrak{g}}$ *is a graded subalgebra of* $\mathbf{A}$ *and*

$$\mathbf{A}^{\mathfrak{g}}[k]=(\mathbf{A}[k])^{\mathfrak{g}}.$$

(ii) *If* $\Psi$ *is a subset of* $R^+$ *such that for all* $\lambda,\mu,\nu\in P^+$ *with* $\lambda\le_\Psi\mu\le_\Psi\nu$, *we have* $d_\Psi(\lambda,\mu)+d_\Psi(\mu,\nu)=d_\Psi(\lambda,\nu)$, *then* $\mathbf{A}^{\mathfrak{g}}_{\Psi}(F)$ *is a graded subalgebra of* $\mathbf{A}^{\mathfrak{g}}$ *for all* $F\subset P^+$.

**1.6.** We denote $\mathbf{T}$ (respectively, $\mathbf{S}$, $\mathbf{E}$) the algebra $\mathbf{A}$ with $A=T(\mathfrak{g})$ (respectively, $A=S(\mathfrak{g})$, $A=\bigwedge\mathfrak{g}$), the $\mathfrak{g}$-module structure on $A$ being given by the usual diagonal action. Our first result is the following.

**Theorem 1.** Let $\xi\in P$ be such that $\Psi=\Psi(\xi)\subset R^+$. Let $A$ be either $S(\mathfrak{g})$ or $\bigwedge\mathfrak{g}$.

(i) Let $\mu,\nu\in P^+$. The subalgebras $\mathbf{A}^{\mathfrak{g}}_{\Psi}(\le_\Psi\nu)$, $\mathbf{A}^{\mathfrak{g}}_{\Psi}(\mu\le_\Psi)$ and $\mathbf{A}^{\mathfrak{g}}_{\Psi}([\mu,\nu]_\Psi)$ of $\mathbf{A}^{\mathfrak{g}}_{\Psi}$ are Koszul. For $A=S(\mathfrak{g})$ all these subalgebras have global dimension at most $|\Psi|$ and the bound is attained for some $\mu',\nu'\in P^+$ with $\mu'\le_\Psi\nu'$.
(ii) The algebra $\mathbf{A}^{\mathfrak{g}}_{\Psi}(P^+)$ is Koszul. Moreover, $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ has left global dimension $|\Psi|$ and its quadratic dual is $(\mathbf{E}^{\mathfrak{g}}_{\Psi})^{op}$.

**Remark.** An alternative characterization of the sets $\Psi(\xi)$ can be found in [6] together with a complete list of such sets for $\mathfrak{g}$ of classical types. In particular, one sees that as the rank of the Lie algebra varies one can find sets $\Psi(\xi)$ of arbitrary cardinality. As a consequence, we see that the Theorem implies the existence of an infinite-dimensional Koszul algebra of global dimension $k$ for any $k\ge 1$.

**1.7.** As an example of our construction, we produce an algebra of left global dimension 2. Assume that $\mathfrak{g}$ is of rank greater than two and that $\mathfrak{g}$ not of type $A$ or $C$. Let $\theta\in R^+$ be the highest root and choose $i_0\in I$ such that $(\theta,\alpha_{i_0})>0$. Then $2\theta-\alpha_{i_0}\in P^+$ and $\Psi(2\theta-\alpha_{i_0})=\{\theta,\theta-\alpha_{i_0}\}$. In this case it can be shown that $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ is isomorphic to an infinite direct sum of copies of the algebra $\mathfrak{B}$ defined as follows. Consider the following translation quiver

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[r] \arrow[d] & (0,3) \arrow[r] \arrow[d] & \cdots \\
 & (1,0) \arrow[r] & (1,1) \arrow[r] \arrow[d] & (1,2) \arrow[r] \arrow[d] & \cdots \\
 & & (2,0) \arrow[r] & (2,1) \arrow[r] \arrow[d] & \cdots \\
 & & & (3,0) \arrow[r] & \cdots
\end{tikzcd}$$

the translation map being $\tau(m,n)=(m-1,n)$, $m,n\in\mathbf{Z}_+$, $m>0$. Then $\mathfrak{B}$ is the quotient of the path algebra of the quiver by the mesh relations. Thus, this algebra is an infinite

dimensional analogue of the Auslander algebra of the path algebra of type $\mathbb{A}_\infty$. Another example of a finite dimensional algebra from this family was already constructed in [7]. Other examples will appear in [10].

**1.8.** In the rest of the section we explain the motivation for the definition of the algebras $\mathbf{A}_\Psi^{\mathfrak g}$ and our strategy for proving Theorem 1. We prove that $\mathbf{S}_\Psi^{\mathfrak g}(\le_\Psi \lambda)$ is isomorphic to the endomorphism algebra of a projective generator of a category of modules for a certain finite-dimensional algebra $\mathfrak g\ltimes\mathfrak g_{\rm ad}$ defined as follows. As a vector space

$$\mathfrak g\ltimes\mathfrak g_{\rm ad}=\mathfrak g\oplus\mathfrak g,$$

and the Lie bracket is given by,

$$[(x,y),(x',y')]=([x,x'],[x,y']+[y,x']).$$

In particular if we identify $\mathfrak g$ (respectively, $\mathfrak g_{\rm ad}$) with the subspace $\{(x,0):x\in\mathfrak g\}$ (respectively, $\{(0,y):y\in\mathfrak g\}$), then $\mathfrak g_{\rm ad}$ is an abelian Lie ideal in $\mathfrak g\ltimes\mathfrak g_{\rm ad}$. Define a $\mathbf Z_+$-grading on $\mathfrak g\ltimes\mathfrak g_{\rm ad}$ by requiring the elements of $\mathfrak g$ to have degree zero and elements of $\mathfrak g_{\rm ad}$ to have degree one. Then the universal enveloping algebra $\mathbf U(\mathfrak g\ltimes\mathfrak g_{\rm ad})$ is a $\mathbf Z_+$-graded algebra and as a trivial consequence of the PBW theorem, there is an isomorphism of vector spaces

$$\mathbf U(\mathfrak g\ltimes\mathfrak g_{\rm ad})\cong S(\mathfrak g)\otimes\mathbf U(\mathfrak g).$$

Our main motivation for the study of $\mathfrak g\ltimes\mathfrak g_{\rm ad}$ as a $\mathbf Z_+$-graded Lie algebra stems from the easy observation that

$$\mathfrak g\ltimes\mathfrak g_{\rm ad}\cong\mathfrak g\otimes\mathbf C[t,t^{-1}]/((t-1)^2)$$

as $\mathbf Z_+$-graded Lie algebras, the right hand side being graded by powers of $(t-1)$.

**1.9.** Let $\mathcal G_2$ be the category whose objects are $\mathbf Z_+$-graded $\mathfrak g\ltimes\mathfrak g_{\rm ad}$-modules with finite-dimensional graded pieces and where the morphisms are $\mathfrak g\ltimes\mathfrak g_{\rm ad}$-module maps which preserve the grading. In other words a $\mathfrak g\ltimes\mathfrak g_{\rm ad}$-module $V$ is an object of $\mathcal G_2$ if and only if

$$V=\bigoplus_{k\in\mathbf Z_+}V[k],\qquad \dim V[k]<\infty,$$
$$\mathfrak gV[k]\subset V[k],\qquad \mathfrak g_{\rm ad}V[k]\subset V[k+1].$$

If $V,W\in\operatorname{Ob}\mathcal G_2$, then

$$\operatorname{Hom}_{\mathcal G_2}(V,W)=\{f\in\operatorname{Hom}_{\mathfrak g\ltimes\mathfrak g_{\rm ad}}(V,W):f(V[k])\subset W[k]\}.$$

We prove that simple objects in this category are parametrized by the set $\Lambda=P^+\times\mathbf Z_+$. For $(\lambda,r)\in\Lambda$ let $V(\lambda,r)$ be an element in the corresponding isomorphism class. We then prove

**Proposition.** *For all $j\ge 0$, $(\mu,s),(\lambda,r)\in\Lambda$,*

$$\operatorname{Ext}^j_{\mathcal G_2}(V(\mu,s),V(\lambda,r))\cong\begin{cases}\operatorname{Hom}_{\mathfrak g}(\bigwedge^j\mathfrak g\otimes V(\mu),V(\lambda)), & j=r-s,\\ 0, & \textit{otherwise.}\end{cases}$$

**1.10.** Suppose now that we have a subset $\Psi$ of $R^+$ as in Theorem 1. Given $\lambda \in P^+$, let $\mathcal{G}_2[\leq_\Psi \lambda]$ be the full subcategory of $\mathcal{G}_2$ consisting of finite-dimensional objects $V \in \mathcal{G}_2$ satisfying the following: if $V(\nu, s)$ is a simple constituent of $V$, then $\nu \leq_\Psi \lambda$ and $s = d_\Psi(\nu, \lambda)$. In particular, $\mathcal{G}_2[\leq_\Psi \lambda]$ has only finitely many simple objects and is a Serre subcategory of $\mathcal{G}_2$.

**Theorem 2.** The category $\mathcal{G}_2[\leq_\Psi \lambda]$ has enough projectives. If $V(\nu_k, s_k)$, $k = 1, 2$ are simple objects in $\mathcal{G}_2[\leq_\Psi \lambda]$ then

$$\operatorname{Ext}^j_{\mathcal{G}_2}(V(\nu_1, s_1), V(\nu_2, s_2)) \cong \operatorname{Ext}^j_{\mathcal{G}_2[\leq_\Psi \lambda]}(V(\nu_1, s_1), V(\nu_2, s_2)).$$

If $P(\leq_\Psi \lambda)$ is a projective generator of $\mathcal{G}_2[\leq_\Psi \lambda]$ we have an isomorphism of $\mathbf{Z}_+$-graded associative algebras

$$\operatorname{End}_{\mathcal{G}_2[\leq_\Psi \lambda]} P(\leq_\Psi \lambda) \cong \mathbf{S}^{\mathfrak{g}}_\Psi(\leq_\Psi \lambda)^{op}$$

and the category $\mathcal{G}_2[\leq_\Psi \lambda]$ is equivalent to the category of left finite-dimensional $\mathbf{S}^{\mathfrak{g}}_\Psi(\leq_\Psi \lambda)$-modules. In particular, $\mathbf{S}^{\mathfrak{g}}_\Psi(\leq_\Psi \lambda)$ is Koszul and $\operatorname{gl.dim} \mathbf{S}^{\mathfrak{g}}_\Psi(\leq_\Psi \lambda) \leq |\Psi|$.

**1.11.** Once Theorem 2 is established, we then prove that $\mathbf{S}^{\mathfrak{g}}_\Psi$ is quadratic and that its Koszul complex is exact. Further we prove that its quadratic dual is ${\mathbf{E}^{\mathfrak{g}}_\Psi}^{op}$ which implies that $\mathbf{E}^{\mathfrak{g}}_\Psi$ is also Koszul. In particular, this also proves that $\mathbf{E}^{\mathfrak{g}}_\Psi(\leq_\Psi \lambda)$ is isomorphic to the Yoneda algebra of $\mathbf{S}^{\mathfrak{g}}_\Psi(\leq_\Psi \lambda)$ which then establishes Theorem 1.

*In the rest of the paper, we will identify the algebra* $\mathbf{A}$ *with* $\mathbf{V}^{\circledast} \otimes A \otimes \mathbf{V}$ *under the natural isomorphism of* $\mathfrak{g}$*-modules. The algebra structure induced on* $\mathbf{V}^{\circledast} \otimes A \otimes \mathbf{V}$ *is given by*

$$(f \otimes a \otimes v)(g \otimes b \otimes w) = g(v) f \otimes ab \otimes w, \qquad a, b \in A,\ f, g \in \mathbf{V}^{\circledast},\ v, w \in \mathbf{V}.$$

## 2. The categories $\mathcal{G}_2$ and $\mathcal{G}_2[\Gamma]$

**2.1.** Given $\alpha \in R$ denote by $\mathfrak{g}_\alpha$ the corresponding root space. The subspaces $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$ are Lie subalgebras of $\mathfrak{g}$. Fix a Chevalley basis $x^\pm_\alpha$, $\alpha \in R^+$, $h_i$, $i \in I$ of $\mathfrak{g}$ and for $\alpha \in R^+$, set $h_\alpha = [x_\alpha, x_{-\alpha}]$.

Let $\mathcal{F}(\mathfrak{g})$ be the category of finite-dimensional $\mathfrak{g}$-modules with the morphisms being maps of $\mathfrak{g}$-modules. We write $\operatorname{Hom}_{\mathfrak{g}}$ for $\operatorname{Hom}_{\mathcal{F}(\mathfrak{g})}$. If $V \in \operatorname{Ob} \mathcal{F}(\mathfrak{g})$, denote

$$V^{\mathfrak{g}} = \{v \in V : \mathfrak{g} v = 0\}$$

the subspace of $\mathfrak{g}$-invariants in $V$. Recall that

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \qquad V_\lambda = \{v \in V : hv = \lambda(h) v,\ \forall\, h \in \mathfrak{h}\}.$$

Set

$$V^+ = \{v \in V \,:\, \mathfrak{n}^+ v = 0\}, \qquad V^+_\lambda = V^+ \cap V_\lambda.$$

The category $\mathcal{F}(\mathfrak{g})$ is semi-simple, i.e. any object in $\mathcal{F}(\mathfrak{g})$ is isomorphic to a direct sum of simple modules.

Given $\lambda \in P^+$ let $v_\lambda \in V(\lambda)$ be such that

$$\mathfrak{n}^+ v_\lambda = 0, \quad h v_\lambda = \lambda(h) v_\lambda, \quad (x^-_{\alpha_i})^{\lambda(h_i)+1} v_\lambda = 0,$$

for all $h \in \mathfrak{h}$, $i \in I$.

We shall use the following standard results in the course of the paper (cf. [15] for (ii)).

**Lemma.** *Let* $\lambda, \mu \in P^+$, $V \in \operatorname{Ob}\mathcal{F}(\mathfrak{g})$. *Then*

(i) $\dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda), V) = \dim V_\lambda^+$,

(ii) *As vector spaces, we have*

$$\operatorname{Hom}_{\mathfrak{g}}(V(\lambda), V \otimes V(\mu)) \cong \{v \in V_{\lambda-\mu} \,:\, (x_{\alpha_i}^+)^{\mu(h_i)+1} v = 0 = (x_{\alpha_i}^-)^{\lambda(h_i)+1} v\}. \qquad \square$$

(iii) *Let* $U, V, W \in \operatorname{Ob}\mathcal{F}(\mathfrak{g})$. *The canonical map* $U^* \otimes V \to \operatorname{Hom}_{\mathbf{C}}(U, V)$ *is an isomorphism of* $\mathfrak{g}$*-modules and we have* $(U^* \otimes V)^{\mathfrak{g}} \cong \operatorname{Hom}_{\mathfrak{g}}(U, V)$. *The natural* $\mathfrak{g}$*-module map* $\operatorname{Hom}_{\mathbf{C}}(U, V) \otimes \operatorname{Hom}_{\mathbf{C}}(V, W) \to \operatorname{Hom}_{\mathbf{C}}(U, W)$ *given by composition and its restriction to* $\operatorname{Hom}_{\mathfrak{g}}(U, V) \otimes \operatorname{Hom}_{\mathfrak{g}}(V, W) \to \operatorname{Hom}_{\mathfrak{g}}(U, W)$ *induces the* $\mathfrak{g}$*-module map* $(U^* \otimes V) \otimes (V^* \otimes W) \to (U^* \otimes W)$ *given by*

$$(f \otimes v) \otimes (v^* \otimes w) \mapsto v^*(v) f \otimes w,$$

*and the restriction* $(U^* \otimes V)^{\mathfrak{g}} \otimes (V^* \otimes W)^{\mathfrak{g}} \to (U^* \otimes W)^{\mathfrak{g}}$

**2.2.** Set $\Lambda = P^+ \times \mathbf{Z}_+$. Given $(\lambda, r) \in \Lambda$, define $V(\lambda, r) \in \operatorname{Ob}\mathcal{G}_2$ by

$$V(\lambda, r)[s] = 0, \qquad r \neq s, \qquad V(\lambda, r)[r] \cong_{\mathfrak{g}} V(\lambda),$$

with $\mathfrak{g}_{\mathrm{ad}} V(\lambda, r) = 0$. Observe that $V(0, 0) \cong \mathbf{C}$ is the trivial $\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ad}}$-module. Set

$$P(\lambda, r) = \mathbf{U}(\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ad}}) \otimes_{\mathbf{U}(\mathfrak{g})} V(\lambda, r). \tag{2.1}$$

Then $P(\lambda, r) \in \operatorname{Ob}\mathcal{G}_2$. It is immediate from the PBW theorem that we have an isomorphism of $\mathbf{Z}_+$-graded $\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ad}}$-modules

$$P(\lambda, r) \cong S(\mathfrak{g}_{\mathrm{ad}}) \otimes V(\lambda, r), \qquad P(\lambda, r)[k] \cong_{\mathfrak{g}} S^{k-r}(\mathfrak{g}_{\mathrm{ad}}) \otimes V(\lambda, r).$$

For $V \in \operatorname{Ob}\mathcal{G}_2$ and $(\lambda, r) \in \Lambda$, set

$$[V : V(\lambda, r)] = \dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda), V[r]).$$

If $V$ is finite-dimensional, then $[V : V(\lambda, r)]$ is just the multiplicity of $V(\lambda, r)$ in a Jordan-Holder series for $V$. We have

$$[P(\lambda, r) : V(\mu, s)] = \dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), S^{s-r}(\mathfrak{g}_{\mathrm{ad}}) \otimes V(\lambda)). \tag{2.2}$$

The next proposition is easily established along the lines of [7, Proposition 2.1]. We include a short sketch of the proof for the reader's convenience.

**Proposition.** *Let* $(\lambda, r) \in \Lambda$.

(i) *The object* $V(\lambda, r)$ *is the unique simple quotient of* $P(\lambda, r)$ *and hence* $P(\lambda, r)$ *is its projective cover in* $\mathcal{G}_2$. *Moreover, the kernel of the canonical morphism* $P(\lambda, r) \to V(\lambda, r)$ *is generated by* $P(\lambda, r)[r+1]$.

(ii) *The isomorphism classes of simple objects in* $\mathcal{G}_2$ *are parametrized by pairs* $(\lambda, r) \in \Lambda$ *and we have*

$$\operatorname{Hom}_{\mathcal{G}_2}(V(\lambda, r), V(\mu, s)) = 0, \qquad (\lambda, r) \neq (\mu, s),$$
$$\operatorname{Hom}_{\mathcal{G}_2}(V(\lambda, r), V(\lambda, r)) \cong \mathbf{C}.$$

(iii) *$P(\lambda, r)$ is the $\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ad}}$-module generated by an element $v_\lambda$ with relations*

$$(\mathfrak{n}^+)v_\lambda = 0, \qquad hv_\lambda = \lambda(h)v_\lambda, \qquad (x^-_{\alpha_i})^{\lambda(h_i)+1}v_\lambda = 0,$$

*for all $h \in \mathfrak{h}$ and $i \in I$.*

(iv) *If $V$ is concentrated in degree $k$ for some $k \in \mathbf{Z}_+$, then $V$ is semi-simple and $P(V) = S(\mathfrak{g}_{\mathrm{ad}}) \otimes V \in \mathrm{Ob}\, \mathcal{G}_2$ is the projective cover of $V$ in $\mathcal{G}_2$. Moreover, if $W$ is also concentrated in degree $k$, then $P(V \oplus W) \cong P(V) \oplus P(W)$.*

*Proof.* It is clear that $V(\lambda, r)$ is simple. To prove that $P(\lambda, r)$ is the projective cover of $V(\lambda, r)$, note that it is indecomposable since it is generated by $P(\lambda, r)[r]$ which is isomorphic to the simple $\mathfrak{g}$-module $V(\lambda)$. The fact that it is projective is standard. To prove (ii), it suffices to note that any simple $\mathcal{G}_2$-module $V$ must satisfy $V[k] \neq 0$ for at most one $k = r \in \mathbf{Z}_+$. In that case, $\mathfrak{g}_{\mathrm{ad}}(V[r]) \subset V[r+1] = 0$ and hence $V[r] \cong V(\lambda)$ for some $\lambda \in P^+$. The other parts of the proposition are now straightforward and we omit the details. □

**2.3.** *From now on we fix $\xi \in P \setminus \{0\}$ and assume that $\Psi = \Psi(\xi) \subset R^+$ where $\Psi(\xi) = \{\alpha \in R : (\alpha, \xi) = \max \xi\}$.* We will use the following property of these sets repeatedly in the course of the paper.

**Lemma.** *Let $\Psi = \Psi(\xi)$. Suppose that*

$$\sum_{\alpha \in R} m_\alpha \alpha = \sum_{\beta \in \Psi} n_\beta \beta, \qquad m_\alpha, n_\beta \in \mathbf{Z}_+.$$

*Then*

$$\sum_{\beta \in \Psi} n_\beta \leq \sum_{\alpha \in R} m_\alpha \tag{2.3}$$

*with equality if and only if $m_\alpha = 0$ for all $\alpha \notin \Psi$.*

*Proof.* We have

$$\max \xi \sum_{\beta \in \Psi} n_\beta = \sum_{\beta \in \Psi} n_\beta (\beta, \xi) = \sum_{\alpha \in R} m_\alpha (\alpha, \xi) \leq \max \xi \sum_{\alpha \in R} m_\alpha.$$

The inequality in (2.3) follows since $\max \xi > 0$, while the equality holds if and only if

$$\sum_{\alpha \in R} m_\alpha (\max \xi - (\alpha, \xi)) = 0.$$

Since $m_\alpha \in \mathbf{Z}_+$ this implies that equality holds if and only if $m_\alpha = 0$ unless $(\alpha, \xi) = \max \xi$. □

**Remark.** In [6] it is shown that the converse of the Lemma is true as well and that in fact if we let $\rho_\xi$ be the sum of elements in $\Psi(\xi)$, then $\Psi(\xi) = \Psi(\rho_\xi)$.

**2.4.** Let $\Gamma$ be a subset of $\Lambda$ and $\mathcal{G}_2[\Gamma]$ be the full subcategory of $\mathcal{G}_2$ consisting of objects $V$ such that
$$[V : V(\mu, s)] \neq 0 \implies (\mu, s) \in \Gamma.$$
Let $V \to V_\Gamma$ be the functor from $\mathcal{G}_2 \to \mathcal{G}_2[\Gamma]$ defined by requiring $V_\Gamma$ to be the maximal $\mathcal{G}_2[\Gamma]$-subobject of $V$. In general one cannot say much about these functors. However, in the next proposition we shall see that for some special choices of $\Gamma$ and $V$ the module $V^\Gamma := V/V_{\Lambda\setminus\Gamma}$ is an object in $\mathcal{G}_2[\Gamma]$.

Given $\mu \leq_\Psi \lambda \in P^+$, let
$$\Lambda(\leq_\Psi \lambda) = \{(\nu, d_\Psi(\nu, \lambda)) : \nu \in P^+,\ \nu \leq_\Psi \lambda\},$$
$$\Lambda([\mu, \lambda]_\Psi) = \{(\nu, d_\Psi(\nu, \lambda)) \,:\, \nu \in P^+,\ \mu \leq_\Psi \nu \leq_\Psi \lambda\}.$$
The set $\Lambda(\leq_\Psi \lambda)$ (and hence $\Lambda([\mu, \lambda]_\Psi)$) is finite and we denote the corresponding category by $\mathcal{G}_2[\leq_\Psi \lambda]$ (respectively, by $\mathcal{G}_2[[\mu, \lambda]_\Psi]$).

**Lemma.** *Let $\lambda' \leq_\Psi \lambda \in P^+$. Set $\Gamma = \Lambda(\leq_\Psi \lambda)$ or $\Gamma = \Lambda([\lambda', \lambda]_\Psi)$ and let $(\mu, r) \in \Gamma$. We have*
$$[P(\mu, r)_{\Lambda\setminus\Gamma} : V(\nu, s)] = \begin{cases} [P(\mu, r) : V(\nu, s)], & (\nu, s) \notin \Gamma, \\ 0, & (\nu, s) \in \Gamma \end{cases}$$
*In particular, $P(\mu, r)^\Gamma$ is an object in $\mathcal{G}_2[\Gamma]$ and $[P(\mu, r)^\Gamma : V(\nu, s)] = [P(\mu, r) : V(\nu, s)]$ for all $(\nu, s) \in \Gamma$.*

*Proof.* Set $P = P(\mu, r)$ and for any subset $\Gamma'$ of $\Lambda$, let $P(\Gamma')$ be the $\mathfrak{g}$-submodule of $P$ generated by elements of the subspace
$$\{v \in P[s]^+_\nu : (\nu, s) \in \Gamma'\}.$$
The Lemma obviously follows if we prove that $P(\Lambda \setminus \Gamma)$ is an object in $\mathcal{G}_2$, i.e. that it is a $\mathbf{Z}_+$-graded $\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ad}}$-module in which case we have $P(\mu, r)_{\Lambda\setminus\Gamma} = P(\Lambda \setminus \Gamma)$. Let $v_\nu \in P[s]^+_\nu$ for some $(\nu, s) \in \Lambda \setminus \Gamma$ and let $V = \mathbf{U}(\mathfrak{g})v_\nu$. Consider the map $\mathfrak{g}_{\mathrm{ad}} \otimes V \to P[s+1]$ given by $x_{\mathrm{ad}} \otimes v \to x_{\mathrm{ad}} v$ and suppose that its image has a non-zero projection onto some $\mathfrak{g}$-module $\mathbf{U}(\mathfrak{g})v_\zeta$, where $v_\zeta \in P[s+1]^+_\zeta$. It suffices to prove that $(\zeta, s+1) \in \Lambda \setminus \Gamma$. Note that either $\zeta = \nu$ or $\zeta = \nu - \beta_0$ for some $\beta_0 \in R$. Assume for a contradiction that $(\zeta, s+1) \in \Gamma$.

Since $(\mu, r), (\zeta, s+1) \in \Gamma$, we can write
$$\lambda - \mu = \sum_{\alpha \in \Psi} n_\alpha \alpha, \qquad \lambda - \zeta = \sum_{\alpha \in \Psi} m_\alpha \alpha,$$
with $\sum_{\alpha\in\Psi} n_\alpha = r$ and $\sum_{\alpha\in\Psi} m_\alpha = s+1$. Moreover since $V \cong_{\mathfrak{g}} V(\nu)$, we have
$$\dim \operatorname{Hom}_{\mathfrak{g}}(V, P(\mu, r)[s]) = [P(\mu, r) : V(\nu, s)] = \dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu), S^{s-r}(\mathfrak{g}_{\mathrm{ad}}) \otimes V(\mu)) \neq 0,$$
which by Lemma 2.1(ii) implies that
$$\nu = \mu - \sum_{\beta \in R} k_\beta \beta, \qquad k_\beta \in \mathbf{Z}_+, \quad \sum_{\beta \in R} k_\beta \leq s - r,$$
which gives
$$\lambda - \nu = \sum_{\alpha \in \Psi} n_\alpha \alpha + \sum_{\beta \in R} k_\beta \beta.$$

If $\zeta = \nu$, we now get by using Lemma 2.3 that

$$\sum_{\alpha\in\Psi} m_\alpha = s+1 \le \sum_{\alpha\in\Psi} n_\alpha + \sum_{\beta\in R} k_\beta \le s$$

which is absurd and hence we must have $\zeta - \nu = \beta_0$ for some $\beta_0 \in R$. This gives

$$\lambda - \zeta = \sum_{\alpha\in\Psi} m_\alpha \alpha = \sum_{\alpha\in\Psi} n_\alpha \alpha + \sum_{\beta\in R} k_\beta \beta + \beta_0,$$

and hence

$$\sum_{\alpha\in\Psi} m_\alpha = \sum_{\alpha\in\Psi} n_\alpha + \sum_{\beta\in R} k_\beta + 1 \le s+1.$$

By Lemma 2.3 again, this means that $k_\beta = 0$ for all $\beta \notin \Psi$ and $\beta_0 \in \Psi$. Therefore $\nu \le_\Psi \zeta \le_\Psi \lambda$ and $d_\Psi(\nu,\lambda) = s$, hence $(\nu, s) \in \Gamma$ which is a contradiction. □

**2.5.** In the following Proposition we gather the properties of projectives in $\mathcal{G}_2[\le_\Psi \lambda]$ that will be needed later.

**Proposition.** *Let $\lambda' \le_\Psi \lambda \in P^+$. Let $\Gamma = \Lambda(\le_\Psi \lambda)$ or $\Gamma = \Lambda([\lambda', \lambda]_\Psi)$.*

(i) *If $V \in \operatorname{Ob}\mathcal{G}_2[\Gamma]$ is concentrated in degree $r$, then $P(V)^\Gamma$ is the projective cover of $V$ in $\mathcal{G}_2[\Gamma]$. In particular, if $(\mu, r) \in \Gamma$, then $P(\mu, r)^\Gamma$ is the projective cover of $V(\mu, r)$ in $\mathcal{G}_2[\Gamma]$ and the category $\mathcal{G}_2[\Gamma]$ has enough projectives.*

(ii) *Suppose that $(\xi, r) \notin \Gamma$ and*

$$\lambda - \xi = \sum_{\alpha\in R} n_\alpha \alpha, \qquad n_\alpha \in \mathbf{Z}_+, \ \sum_{\alpha\in R} n_\alpha \le r.$$

*Then $P(\xi, r) \in \operatorname{Ob}\mathcal{G}_2[\Lambda \setminus \Gamma]$.*

(iii) *For $(\mu, r), (\nu, s) \in \Gamma$, we have*

$$[P(\mu,r)^\Gamma : V(\nu,s)] = \dim \operatorname{Hom}_{\mathcal{G}_2}(P(\nu,s)^\Gamma, P(\mu,r)^\Gamma) \ne 0,$$

*only if $\nu \le_\Psi \mu$ and $d_\Psi(\nu,\mu) = s - r$.*

(iv) $\operatorname{Hom}_{\mathcal{G}_2}(P(\nu,s), P(\mu,r)) \cong \operatorname{Hom}_{\mathcal{G}_2}(P(\nu,s)^\Gamma, P(\mu,r)^\Gamma)$ *and this isomorphism is compatible with compositions.*

*Proof.* It follows from Lemma 2.4 that if $M \in \operatorname{Ob}\mathcal{G}_2[\Gamma]$ and $(\mu, r) \in \Gamma$, then

$$\operatorname{Hom}_{\mathcal{G}_2}(P(\mu,r)_{\Lambda\setminus\Gamma}, M) = 0,$$

or equivalently that

$$\operatorname{Hom}_{\mathcal{G}_2}(P(\mu,r)^\Gamma, M) \cong \operatorname{Hom}_{\mathcal{G}_2}(P(\mu,r), M). \tag{2.4}$$

In particular, $\operatorname{Hom}_{\mathcal{G}_2}(P(\mu,r)^\Gamma, -)$ is exact on $\mathcal{G}_2[\Gamma]$ and hence $P(\mu,r)^\Gamma$ is projective. The fact that it is isomorphic to the projective cover of $V(\mu,r)$ in $\mathcal{G}_2[\Gamma]$ is immediate.

To prove (ii), suppose that $[P(\xi,r) : V(\nu,s)] \ne 0$ for some $(\nu,s) \in \Gamma$. By (2.2)

$$[P(\xi,r) : V(\nu,s)] = \dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu), S^{s-r}(\mathfrak{g}_{\mathrm{ad}}) \otimes V(\xi)),$$

hence by Lemma 2.1(ii)

$$\xi - \nu = \sum_{\alpha\in R} m_\alpha \alpha, \qquad m_\alpha \in \mathbf{Z}_+, \ \sum_{\alpha\in R} m_\alpha \le s - r.$$

Using Lemma 2.3 we conclude that $\nu \le_\Psi \xi \le_\Psi \lambda$, and $d_\Psi(\xi,\lambda) = r$. This forces $(\xi, r) \in \Gamma$ which is a contradiction.

To prove part (iii), note that (2.2) and Lemma 2.4 imply that

$$[P(\mu,r)^\Gamma : V(\nu,s)] = [P(\mu,r) : V(\nu,s)] = \dim \operatorname{Hom}_{\mathfrak g}(V(\nu), S^{s-r}(\mathfrak g_{\rm ad}) \otimes V(\mu)).$$

In particular, if $[P(\mu,r)^\Gamma : V(\nu,s)] \neq 0$, we must have

$$\mu - \nu = \sum_{\alpha\in R} k_\alpha \alpha, \qquad k_\alpha \in \mathbf Z_+,$$

and $0 \le \sum_{\alpha\in R} k_\alpha \le s-r$. Since $(\mu,r)\in\Gamma$ we write $\lambda - \mu = \sum_{\alpha\in\Psi} \ell_\alpha \alpha$ for some $\ell_\alpha \in \mathbf Z_+$ with $\sum_{\alpha\in\Psi} \ell_\alpha = r$. Then

$$\lambda - \nu = \sum_{\alpha\in\Psi} \ell_\alpha\alpha + \sum_{\alpha\in R} k_\alpha \alpha, \qquad \sum_{\alpha\in\Psi}\ell_\alpha + \sum_{\alpha\in R} k_\alpha \le s.$$

Since $(\nu, s) \in \Gamma$ it follows from Lemma 2.3 that

$$\sum_{\alpha\in\Psi}\ell_\alpha + \sum_{\alpha\in R} k_\alpha = s,$$

hence $k_\alpha = 0$ if $\alpha \notin \Psi$. This implies that $\mu \le_\Psi \nu$ and

$$\sum_{\alpha\in R} k_\alpha = \sum_{\alpha\in\Psi} k_\alpha = s - r,$$

hence $d_\Psi(\mu,\nu) = s-r$. The equality in (iii) is a standard property of projectives.

To prove part (iv), note that (2.4) implies

$$\operatorname{Hom}_{\mathcal G_2}(P(\nu,s)^\Gamma, P(\mu,r)^\Gamma) \cong \operatorname{Hom}_{\mathcal G_2}(P(\nu,s), P(\mu,r)^\Gamma).$$

Since $\operatorname{Hom}_{\mathcal G_2}(P(\nu,s), P(\mu,r))$ maps onto $\operatorname{Hom}_{\mathcal G_2}(P(\nu,s), P(\mu,r)^\Gamma)$, part (iv) now follows by using Proposition 2.2 and Lemma 2.4, which give

$$\dim \operatorname{Hom}_{\mathcal G_2}(P(\nu,s)^\Gamma, P(\mu,r)^\Gamma) = \dim \operatorname{Hom}_{\mathcal G_2}(P(\nu,s), P(\mu,r)). \qquad \square$$

**2.6.** We can now prove one part of Theorem 2.

**Proposition.** *Assume that $\Gamma = \Lambda(\le_\Psi \lambda)$. The category $\mathcal G_2[\Gamma]$ is equivalent to the category of right finite dimensional modules for the associative algebra $\operatorname{End}_{\mathcal G_2[\Gamma]} P(\Gamma)$, where $P(\Gamma) = \bigoplus_{(\mu,s)\in\Gamma} P(\mu,s)^\Gamma$. Moreover, if we set*

$$(\operatorname{End}_{\mathcal G_2[\Gamma]} P(\Gamma))[k] = \bigoplus_{(\mu,r),(\nu,s)\in\Gamma: r-s=k} \operatorname{Hom}_{\mathcal G_2[\Gamma]}(P(\mu,r)^\Gamma, P(\nu,s)^\Gamma), \tag{2.5}$$

*then we have an isomorphism of $\mathbf Z_+$-graded associative algebras*

$$\operatorname{End}_{\mathcal G_2[\Gamma]} P(\Gamma) \cong \mathbf S^{\mathfrak g}_\Psi(\le_\Psi \lambda)^{op}.$$

*In particular, $\mathcal G_2[\Gamma]$ is equivalent to the category of left finite dimensional $\mathbf S^{\mathfrak g}_\Psi(\le_\Psi \lambda)$-modules.*

*The corresponding results also hold for $\lambda' \le_\Psi \lambda$, $\Gamma = \Lambda([\lambda',\lambda]_\Psi)$ and $\mathbf S^{\mathfrak g}_\Psi([\lambda',\lambda]_\Psi)$.*

*Proof.* By Proposition 2.5(i), $P(\Gamma)$ is a projective generator of $\mathcal{G}_2[\Gamma]$. The equivalence of categories is then standard and is provided by the extact functor $\mathrm{Hom}_{\mathcal{G}_2}(P(\Gamma), -)$. To prove the second assertion, note that by Proposition 2.5(iii), (2.5) defines a grading on $\mathrm{End}_{\mathcal{G}_2} P(\Gamma)$. Then by Proposition 2.5(iv) we have

$$\mathrm{End}_{\mathcal{G}_2} P(\Gamma) \cong \bigoplus_{(\mu,r),(\nu,s)\in\Gamma} \mathrm{Hom}_{\mathcal{G}_2}(P(\mu,r), P(\nu,s))$$

as $\mathbf{Z}_+$-graded associative algebras. Furthermore, observe that there exists a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{G}_2}(P(\mu,r), P(\nu,s)) \cong \mathrm{Hom}_{\mathfrak{g}}(V(\mu), S^{r-s}(\mathfrak{g}_{\mathrm{ad}}) \otimes V(\nu))$$

given by restriction and moreover this map is compatible with compositions, in the sense that if $f \in \mathrm{Hom}_{\mathcal{G}_2}(P(\mu,r), P(\nu,s))$, $g \in \mathrm{Hom}_{\mathcal{G}_2}(P(\nu,s), P(\xi,k))$, then the following diagram commutes

$$\begin{tikzcd}
 & S^{r-s}(\mathfrak{g}) \otimes V(\nu) \arrow[dr, "1\otimes g|_{1\otimes V(\nu,s)}"] & \\
,\ V(\mu) \arrow[ur, "f|_{1\otimes V(\mu,r)}"] \arrow[dr, "(g\circ f)|_{1\otimes V(\mu,r)}"'] & & S^{r-s}(\mathfrak{g}) \otimes S^{s-k}(\mathfrak{g}) \otimes V(\nu) \arrow[dl, "m_{S(\mathfrak{g})}\otimes 1"] \\
 & S^{r-k}(\mathfrak{g}) \otimes V(\xi) &
\end{tikzcd}$$

where $m_{S(\mathfrak{g})} : S(\mathfrak{g}) \otimes S(\mathfrak{g}) \to S(\mathfrak{g})$ is the multiplication map. The proposition now follows from Lemma 2.1(iii) along with the observation that the multiplication of $\mathbf{S}^{\mathfrak{g}}$ when restricted to $1_\mu \mathbf{S}^{\mathfrak{g}}[r-s]1_\nu \otimes 1_\nu \mathbf{S}^{\mathfrak{g}}[s-k]1_\xi$ is just the natural map

$$(V(\mu)^* \otimes S^{r-s}(\mathfrak{g}) \otimes V(\nu))^{\mathfrak{g}} \otimes (V(\nu)^* \otimes S^{s-k}(\mathfrak{g}) \otimes V(\xi)) \to (V(\mu)^* \otimes S^{r-s}(\mathfrak{g}) \otimes S^{s-k}(\mathfrak{g}) \otimes V(\xi))^{\mathfrak{g}}$$

composed with $1 \otimes m_{S(\mathfrak{g})} \otimes 1$. ☐

## 3. Homological properties of category $\mathcal{G}_2$

**3.1.** Given $(\mu, r) \in \Lambda$ and $0 \le j \le \dim \mathfrak{g}$, set

$$P_j(\mu,r) = S(\mathfrak{g}_{\mathrm{ad}}) \otimes (\textstyle\bigwedge^j \mathfrak{g}_{\mathrm{ad}} \otimes V(\mu))[j+r],$$

where we regard $(\bigwedge^j \mathfrak{g}_{\mathrm{ad}} \otimes V(\mu))[j+r]$ as a $\mathfrak{g} \ltimes \mathfrak{g}_{\mathrm{ad}}$-module concentrated in degree $(j+r)$ with the canonical $\mathfrak{g}$-action. In particular, $P_0(\mu,r) = P(\mu,r)$ and the modules $P_j(\mu,r)$ are projective in $\mathcal{G}_2$.

For $j \in \mathbf{Z}_+$ with $j > 0$, let $d_j : P_j(\mu,r) \to P_{j-1}(\mu,r)$ be the linear map obtained by extending

$$d_j(u \otimes (x_1 \wedge x_2 \wedge \cdots \wedge x_j) \otimes v) = \sum_{s=1}^{j} (-1)^{s-1} u x_s \otimes (x_1 \wedge \cdots \wedge x_{s-1} \wedge x_{s+1} \wedge \cdots \wedge x_j) \otimes v,$$

and let $d_0 : P_0(\mu, r) \to V(\mu, r)$ be

$$d_0(u \otimes v) = uv$$

where $u \in S(\mathfrak{g}_{\rm ad})$, $v \in V(\mu, r)$ and $x_s \in \mathfrak{g}$ for $1 \le s \le j$.

**Proposition.** *Let $(\mu, r) \in \Lambda$, $N = \dim \mathfrak{g}$. The sequence*

$$0 \longrightarrow P_N(\mu, r) \xrightarrow{d_N} P_{N-1}(\mu, r) \xrightarrow{d_{N-1}} \cdots \xrightarrow{d_2} P_1(\mu, r) \xrightarrow{d_1} P(\mu, r) \xrightarrow{d_0} V(\mu, r) \longrightarrow 0$$

*is a projective resolution of $V(\mu, r)$ in the category $\mathcal{G}_2$.*

*Proof.* To prove the exactness, note that the above sequence is obtained by tensoring the Koszul complex for $S(\mathfrak{g})$ with $V(\mu, r)$ and introducing an appropriate grading. The map $d_j$ is just $D \otimes 1$ where $D$ is the Koszul differential. The exactness is then standard (see for example [12]). It is straightforward to check that $d_j$ is a morphism in $\mathcal{G}_2$ for all $j$, which proves the proposition. □

**Corollary.** *Let $j \ge 0$. We have*

$$d_j(u \otimes a \otimes v) = (u \otimes 1) d_j(1 \otimes a \otimes v), \qquad \forall\, u \in S(\mathfrak{g}_{\rm ad}),\ a \in \textstyle\bigwedge^j \mathfrak{g}_{\rm ad},\ v \in V(\mu, r).$$

*In particular* $\operatorname{Im} d_j[s] = 0$ *if* $j + r > s$ *and*

$$\operatorname{Im} d_{s-r}[s] \cong_{\mathfrak{g}} P_{s-r}(\mu, r)[s] \cong_{\mathfrak{g}} \textstyle\bigwedge^{s-r} \mathfrak{g}_{\rm ad} \otimes V(\mu).$$

*Proof.* The first two assertions are immediate. To prove the last assertion, observe that since $\ker d_{s-r}[s] = 0 = \operatorname{Im} d_{s-r+1}[s]$, we get $d_{s-r}(P_{s-r}(\mu, r)[s]) \cong_{\mathfrak{g}} P_{s-r}(\mu, r)[s]$. □

**3.2.** We can now compute the Ext spaces for all simple objects in the category $\mathcal{G}_2$.

**Proposition.** *For all $j \ge 0$, $(\mu, r), (\nu, s) \in \Lambda$,*

$$\operatorname{Ext}^j_{\mathcal{G}_2}(V(\mu, r), V(\nu, s)) \cong \begin{cases} \operatorname{Hom}_{\mathfrak{g}}(\bigwedge^j \mathfrak{g}_{\rm ad} \otimes V(\mu), V(\nu)), & j = s - r, \\ 0, & \textit{otherwise.} \end{cases}$$

*Proof.* For $j \ge 1$, the short exact sequence

$$0 \to \operatorname{Im} d_j \to P_{j-1}(\mu, r) \to \operatorname{Im} d_{j-1} \to 0,$$

yields the isomorphism,

$$\operatorname{Ext}^j_{\mathcal{G}_2}(V(\mu, r), V(\nu, s)) \cong \operatorname{Ext}^1_{\mathcal{G}_2}(\operatorname{Im} d_{j-1}, V(\nu, s)),$$

and also the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{G}_2}(\operatorname{Im} d_{j-1}, V(\nu, s)) \to \operatorname{Hom}_{\mathcal{G}_2}(P_{j-1}(\mu, r), V(\nu, s)) \to \\ \operatorname{Hom}_{\mathcal{G}_2}(\operatorname{Im} d_j, V(\nu, s)) \to \operatorname{Ext}^1_{\mathcal{G}_2}(\operatorname{Im} d_{j-1}, V(\nu, s)) \to 0.$$

We claim that $\operatorname{Hom}_{\mathcal{G}_2}(\operatorname{Im} d_j, V(\nu, s)) = 0$ unless $j = s - r$ which proves that

$$\operatorname{Ext}^j_{\mathcal{G}_2}(V(\mu, r), V(\nu, s)) \cong \operatorname{Ext}^1_{\mathcal{G}_2}(\operatorname{Im} d_{j-1}, V(\nu, s)) = 0, \quad j \ne s - r.$$

To prove the claim, note that by Corollary 3.1 we may assume $j + r \le s$ and also that if $v \in \operatorname{Im} d_j[s]$, then

$$v = \sum_p (u_p \otimes 1) d_j (1 \otimes w_p),$$

where $u_p \in S^{s-r-j}(\mathfrak{g}_{\rm ad})$ and $w_p \in (\bigwedge^j \mathfrak{g}_{\rm ad} \otimes V(\mu))[j + r]$. Suppose first that $j + r < s$. Let $f \in \operatorname{Hom}_{\mathcal{G}_2}(\operatorname{Im} d_j, V(\nu, s))$. Then $f(1 \otimes w_p) = 0$ and so $f(v) = \sum_p u_p f(w_p) = 0$, that is $f = 0$.

Suppose that $j = s - r$. Since $P_{s-r-1}(\mu, r)$ is the projective cover of a semi-simple $\mathfrak{g} \ltimes \mathfrak{g}_{\rm ad}$-module concentrated in degree $s - 1$, it follows that

$$\operatorname{Hom}_{\mathcal{G}_2}(P_{s-r-1}(\mu, r), V(\nu, s)) = 0,$$

and hence

$$\begin{aligned}\operatorname{Ext}^{s-r}_{\mathcal{G}_2}(V(\mu, r), V(\nu, s)) &\cong \operatorname{Ext}^1_{\mathcal{G}_2}(\operatorname{Im} d_{s-r-1}, V(\nu, s)) \cong \operatorname{Hom}_{\mathcal{G}_2}(\operatorname{Im} d_{s-r}, V(\nu, s))\\ &\cong \operatorname{Hom}_{\mathfrak{g}}(\operatorname{Im} d_{s-r}[s], V(\lambda)).\end{aligned}$$

The result follows by applying Corollary 3.1. $\square$

**3.3.**

**Proposition.** *Let $\lambda' \le_\Psi \lambda \in P^+$. Assume that $\Gamma = \Lambda(\le_\Psi \lambda)$ or $\Gamma = \Lambda([\lambda', \lambda]_\Psi)$ Let $(\mu, r) \in \Gamma$. Then the induced map $d_j^\Gamma : P_j(\mu, r)^\Gamma \to P_j(\mu, r-1)^\Gamma$ is a morphism of objects in $\mathcal{G}_2[\Gamma]$ and the sequence*

$$0 \longrightarrow P_N(\mu, r)^\Gamma \xrightarrow{d_N^\Gamma} P_{N-1}(\mu, r)^\Gamma \xrightarrow{d_{N-1}^\Gamma} \cdots \xrightarrow{d_2^\Gamma} P_1(\mu, r)^\Gamma \xrightarrow{d_1^\Gamma} P(\mu, r)^\Gamma \xrightarrow{d_0^\Gamma} V(\mu, r) \longrightarrow 0$$

*is a projective resolution of $V(\mu, r)$ in the category $\mathcal{G}_2[\Gamma]$. In particular for all $(\nu, s) \in \Gamma$ we have*

$$\operatorname{Ext}^j_{\mathcal{G}_2[\Gamma]}(V(\mu, r), V(\nu, s)) \cong \operatorname{Ext}^j_{\mathcal{G}_2}(V(\mu, r), V(\nu, s))$$

*and hence*

$$\operatorname{Ext}^j_{\mathcal{G}_2[\Gamma]}(V(\mu, r), V(\nu, s)) = \begin{cases}\operatorname{Hom}_{\mathfrak{g}}(\bigwedge^j \mathfrak{g}_{\rm ad} \otimes V(\mu), V(\nu)), & \nu \le_\Psi \mu,\ j = d_\Psi(\nu, \mu) = s - r\\ 0, & \textit{otherwise.}\end{cases}$$

*Proof.* Set

$$W_j(\mu, r) = (\textstyle\bigwedge^j \mathfrak{g}_{\rm ad} \otimes V(\mu))[r + j].$$

Since $W_j(\mu, r) \in \operatorname{Ob} \mathcal{G}_2$ is concentrated in degree $r + j$ and is $\mathfrak{g}$-semisimple, it is also a semi-simple object of $\mathcal{G}_2$ and so, we have a $\mathfrak{g} \ltimes \mathfrak{g}_{\rm ad}$ decomposition,

$$W_j(\mu, r) = W_j(\mu, r)_{\Lambda \setminus \Gamma} \oplus W_j(\mu, r)^\Gamma$$

and hence by Proposition 2.2(iv) we have

$$P_j(\mu, r) \cong_{\mathcal{G}_2} P(W_j(\mu, r)) \cong_{\mathcal{G}_2} P(W_j(\mu, r)_{\Lambda \setminus \Gamma}) \oplus P(W_j(\mu, r)^\Gamma).$$

Note that $[W_j(\mu, r) : V(\xi, j + r)] \ne 0$ implies that

$$\mu - \xi = \sum_{\alpha \in R} m_\alpha \alpha, \qquad m_\alpha \in \mathbf{Z}_+, \quad \sum_{\alpha \in R} m_\alpha \le j,$$

and so

$$\lambda-\xi=\sum_{\alpha\in R} m'_\alpha\alpha,\qquad m'_\alpha\in\mathbf{Z}_+,\quad \sum_{\alpha\in R} m'_\alpha\le r+j.$$

It now follows from Proposition 2.5(ii) that $P(W_j(\mu,r)_{\Lambda\setminus\Gamma})\in\operatorname{Ob}\mathcal{G}_2[\Lambda\setminus\Gamma]$, hence

$$P_j(\mu,r)_{\Lambda\setminus\Gamma}\cong_{\mathcal{G}_2} P(W_j(\mu,r)_{\Lambda\setminus\Gamma})\oplus P(W_j(\mu,r)^\Gamma)_{\Lambda\setminus\Gamma}.$$

By Lemma 2.4, $P_j(\mu,r)_{\Lambda\setminus\Gamma}\in\operatorname{Ob}\mathcal{G}_2[\Lambda\setminus\Gamma]$ and so $P_j(\mu,r)^\Gamma$ is a projective object in $\mathcal{G}_2[\Gamma]$ and is isomorphic to $P(W_j(\mu,r)^\Gamma)^\Gamma$. All assertions are now straightforward. $\square$

**3.4.** Let $\lambda_\Psi=\sum_{\beta\in\Psi}\beta$.

**Lemma.** (i) *If* $\mu,\mu+\lambda_\Psi\in P^+$, $\dim\operatorname{Hom}_{\mathfrak g}(V(\mu+\lambda_\Psi),\bigwedge^{|\Psi|}\mathfrak g_{\rm ad}\otimes V(\mu))\le 1$.

(ii) *There exists* $\mu\in P^+$ *such that* $\mu+\lambda_\Psi\in P^+$ *and* $\dim\operatorname{Hom}_{\mathfrak g}(V(\mu+\lambda_\Psi),\bigwedge^{|\Psi|}\mathfrak g_{\rm ad}\otimes V(\mu))=1$.

*Proof.* Using Lemma 2.3 we see that $(\bigwedge^{|\Psi|}\mathfrak g_{\rm ad})_{\lambda_\Psi}=\mathbf{C}\bigwedge_{\beta\in\Psi}x^+_\beta$ and hence $\dim(\bigwedge^{|\Psi|}\mathfrak g_{\rm ad})_{\lambda_\Psi}=1$. Part (i) is now immediate from Lemma 2.1(ii). To prove (ii), choose $k_i\in\mathbf{Z}_+$, $i\in I$ such that

$$(\operatorname{ad}x^+_{\alpha_i})^{k_i+1}{\textstyle\bigwedge_{\beta\in\Psi}}x^+_\beta=0=(\operatorname{ad}x^-_{\alpha_i})^{k_i+1+\lambda_\Psi(h_i)}{\textstyle\bigwedge_{\beta\in\Psi}}x^+_\beta,\quad \lambda_\Psi+\sum_{i\in I}k_i\omega_i\in P^+$$

and set $\mu=\sum_{i\in I}k_i\omega_i$. It follows from Lemma 2.1(ii) that

$$\operatorname{Hom}_{\mathfrak g}(V(\mu+\lambda_\Psi),{\textstyle\bigwedge^{|\Psi|}}\mathfrak g_{\rm ad}\otimes V(\mu))\cong\mathbf{C}{\textstyle\bigwedge_{\beta\in\Psi}}x^+_\beta.\qquad\square$$

**3.5.** Recall that for an abelian category $\mathcal C$ which has enough injectives or projectives, the global dimension, written $\operatorname{gl.dim}\mathcal C$, equals the minimal $j$ such that $\operatorname{Ext}^j_{\mathcal C}(M,N)=0$ for all $M,N\in\operatorname{Ob}\mathcal C$. Note that categories $\mathcal G_2[\le_\Psi\lambda]$, $\lambda\in P^+$ and $\mathcal G_2[[\lambda',\lambda]_\Psi]$, $\lambda'\le_\Psi\lambda$ have enough projectives by Proposition 2.4. We can now prove

**Theorem 3.** Let $\lambda'\le_\Psi\lambda\in P^+$. We have

$$\operatorname{gl.dim}\mathcal G_2[[\lambda',\lambda]_\Psi],\operatorname{gl.dim}\mathcal G_2[\le_\Psi\lambda]\le|\Psi|$$

and the upper bound is attained for some $\lambda'\le_\Psi\lambda\in P^+$. In particular, the algebras $\mathbf S^{\mathfrak g}_\Psi(\le_\Psi\lambda)$, $\mathbf S^{\mathfrak g}_\Psi([\lambda',\lambda]_\Psi)$ have global dimension at most $|\Psi|$ and the upper bound is attained for some $\lambda'\le_\Psi\lambda\in P^+$.

*Proof.* Let $\Gamma=\Lambda(\le_\Psi\lambda)$. Since all objects in $\mathcal G_2[\Gamma]$ have finite length, to establish an upper bound on the global dimension of $\mathcal G_2[\Gamma]$, it suffices to prove that for all $(\mu,r),(\nu,k)\in\Gamma$ we have

$$\operatorname{Ext}^j_{\mathcal G_2[\Gamma]}(V(\mu,r),V(\nu,k))=0,\qquad\forall\, j>|\Psi|.$$

By Proposition 3.2, this amounts to proving that for $\nu\le_\Psi\mu$, $j=d_\Psi(\nu,\mu)$

$$\operatorname{Hom}_{\mathfrak g}({\textstyle\bigwedge^j}\mathfrak g_{\rm ad}\otimes V(\mu),V(\nu))\ne 0\implies|\Psi|\ge j.$$

A weight of $\bigwedge^j\mathfrak g$ is the sum of at most $j$ distinct roots. Hence by Lemma 2.1(ii) it follows that

$$\operatorname{Hom}_{\mathfrak g}({\textstyle\bigwedge^j}\mathfrak g_{\rm ad}\otimes V(\mu),V(\nu))\ne 0\tag{3.1}$$

only if

$$\mu - \nu = \sum_{\alpha \in R} m_\alpha \alpha, \qquad m_\alpha \in \{0,1\}, \quad \sum_{\alpha \in R} m_\alpha \le j.$$

Applying Lemma 2.3, we immediately conclude that $m_\alpha = 0$ unless $\alpha \in \Psi$ and also that $\sum_{\alpha\in\Psi} m_\alpha = j$. Since $m_\alpha \in \{0,1\}$, it follows that $|\Psi| \ge j$.

To prove that the upper bound is attained, observe that by Lemma 3.4(ii), there exists $\mu \in P^+$ such that $\mu + \lambda_\Psi \in P^+$ and $\operatorname{Hom}_{\mathfrak g}(V(\mu+\lambda_\Psi), \bigwedge^{|\Psi|}\mathfrak g_{\rm ad} \otimes V(\mu)) \neq 0$. Then $(\mu, |\Psi|) \in \Lambda(\mu+\lambda_\Psi, \Psi)$ and we have

$$\begin{aligned}\dim \operatorname{Ext}^{|\Psi|}_{\mathcal G_2}(V(\mu+\lambda_\Psi, 0), V(\mu, |\Psi|)) &= \dim \operatorname{Hom}_{\mathfrak g}(\textstyle\bigwedge^{|\Psi|}\mathfrak g_{\rm ad} \otimes V(\mu+\lambda_\Psi), V(\mu))\\ &= \dim \operatorname{Hom}_{\mathfrak g}(V(\mu+\lambda_\Psi), \textstyle\bigwedge^{|\Psi|}\mathfrak g_{\rm ad} \otimes V(\mu)) = 1. \qquad \square\end{aligned}$$

**3.6.**

**Proposition.** *The algebras $\mathbf S^{\mathfrak g}_\Psi$, $\mathbf S^{\mathfrak g}_\Psi(\lambda \le_\Psi)$ have left global dimension $|\Psi|$.*

*Proof.* Given $\mu \in P^+$, let $S_\mu$ be the left simple $\mathbf S^{\mathfrak g}_\Psi$-module $S_\mu$ corresponding to the idempotent $1_\mu$. Then its projective cover in the category $\mathbf S^{\mathfrak g}_\Psi - \operatorname{mod}_f$ of finite dimensional left $\mathbf S^{\mathfrak g}_\Psi$-modules is

$$P_\mu = \mathbf S^{\mathfrak g}_\Psi 1_\mu = \bigoplus_{\nu \le_\Psi \mu} 1_\nu \mathbf S^{\mathfrak g}_\Psi 1_\mu = \mathbf S^{\mathfrak g}_\Psi(\le_\Psi \mu) 1_\mu.$$

Similarly, if $\nu \le_\Psi \mu$, $P_\nu = \mathbf S^{\mathfrak g}_\Psi(\le_\Psi \mu)1_\nu$. In particular, $[P_\mu : S_\nu] = 0$ unless $\nu \le_\Psi \mu$. Proceeding by induction we conclude that $S_\mu$ has a projective resolution in $\mathbf S^{\mathfrak g}_\Psi - \operatorname{mod}_f$

$$\cdots \to P^1_\mu \to P_\mu \to S_\mu \to 0$$

in which $[P^j_\mu : S_\xi] = 0$ unless $\xi \le_\Psi \mu$. Therefore, this projective resolution can be regarded as a projective resolution for $S_\mu$ in the category $\mathbf S^{\mathfrak g}_\Psi(\le_\Psi \mu) - \operatorname{mod}_f$, which is equivalent to the category $\mathcal G_2[\le_\Psi \mu]$ by Proposition 2.6, and $\operatorname{Ext}^j_{\mathbf S^{\mathfrak g}_\Psi - \operatorname{mod}_f}(S_\mu, S_\nu) = 0$ unless $\nu \le_\Psi \mu$. Thus, we have

$$\begin{aligned}\dim \operatorname{Ext}^j_{\mathbf S^{\mathfrak g}_\Psi - \operatorname{mod}_f}(S_\mu, S_\nu) &= \dim \operatorname{Ext}^j_{\mathbf S^{\mathfrak g}_\Psi(\le_\Psi \mu) - \operatorname{mod}_f}(S_\mu, S_\nu)\\ &= \dim \operatorname{Ext}^j_{\mathcal G_2[\le_\Psi \mu]}(V(\mu,0), V(\nu, d_\Psi(\nu,\mu))),\end{aligned}$$

and the result follows from Theorem 3.

The argument for $\mathbf S^{\mathfrak g}_\Psi(\lambda \le_\Psi)$ is similar, with $\mathbf S^{\mathfrak g}_\Psi(\le_\Psi \mu)$ replaced by $\mathbf S^{\mathfrak g}_\Psi([\lambda,\mu]_\Psi)$. $\square$

**3.7.** Let $A = \bigoplus_{r \in \mathbf Z_+} A[r]$ be a $\mathbf Z_+$-graded associative algebra such that $A[0]$ is semi-simple. We can regard $A[0]$ as a graded left $A$-module concentrated in degree 0 via the canonical projection $A \to A/\bigoplus_{j>0} A[j] \cong A[0]$. Following [2, Definition 1.2.1], a given grading on $A$ is said to be Koszul if $A[0]$ admits a projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to A[0] \to 0$$

in the category of left $\mathbf Z_+$-graded $A$-modules such that $P^r$ is generated by $P^r[r]$ as a graded $A$-module.

Suppose that $\dim A[r] < \infty$ for all $r \in \mathbf{Z}_+$ and $A[0]$ is commutative, so that in particular we have $A[0] = \bigoplus_{\mu \in F} \mathbf{C}1_\mu$, where the $1_\mu$ are pairwise orthogonal idempotents and $F$ is a finite index set. Then we have the following numerical criterion for Koszulity ([2, Theorem 2.11.1]). Let $S_\mu = \mathbf{C}1_\mu$ be the simple left $A$-module corresponding to $1_\mu$. Set

$$H(A,t) = (H(A,t)_{\mu,\nu})_{\mu,\nu \in F}, \quad H(E(A),t) = (H(E(A),t)_{\mu,\nu})_{\mu,\nu \in F}$$

where

$$\begin{aligned} H(A,t)_{\mu,\nu} &= \sum_{i \geq 0} t^i \dim(1_\mu A[i] 1_\nu), \\ H(E(A),t)_{\mu,\nu} &= \sum_{i \geq 0} t^i \dim \operatorname{Ext}^i_{A-\mathrm{mod}_f}(S_\nu, S_\mu) \end{aligned}$$

are formal power series in $t$ with coefficients in $\mathbf{Z}_+$. The matrix $H(A,t)$ is the Hilbert matrix of $A$, while $H(E(A),t)$ is the Hilbert matrix of the Yoneda algebra of $A$. The algebra $A$ is Koszul if and only if $H(E(A),-t)$ is the inverse of $H(A,t)$.

**3.8.** The following proposition completes the proof of Theorem 2.

**Proposition.** *Let $\lambda' \leq_\Psi \lambda \in P^+$. The natural grading on $\mathbf{S}^{\mathfrak{g}}_\Psi(\leq_\Psi \lambda)$ and $\mathbf{S}^{\mathfrak{g}}_\Psi([\lambda',\lambda]_\Psi)$ is Koszul.*

*Proof.* Let $F = (\leq_\Psi \lambda)$ or $F = [\lambda',\lambda]_\Psi$ and set $A = \mathbf{S}^{\mathfrak{g}}_\Psi(F)$. For $\mu \in F$ let $S_\mu$ be the simple left $A$-module corresponding to the idempotent $1_\mu$ and let $P_\mu = A1_\mu$ be its projective cover. Then $[P_\mu : S_\nu] = \dim(1_\nu A 1_\mu)$.

By Proposition 2.6, the category $A - \mathrm{mod}_f$ is equivalent to the category $\mathcal{G}_2[F]$. Proposition 2.5 gives

$$\dim(1_\mu A[j] 1_\nu) \neq 0 \implies \mu \leq_\Psi \nu,\ j = d_\Psi(\mu,\nu).$$

and using Proposition 3.3 we also have

$$\operatorname{Ext}^j_{A-\mathrm{mod}_f}(S_\nu, S_\mu) \neq 0 \implies \mu \leq_\Psi \nu,\ j = d_\Psi(\mu,\nu).$$

This shows that the matrices $H(A,t)$ and $H(E(A),t)$, and so their product, are all upper triangular. Moreover, for $\mu \leq_\Psi \nu \in F$ we have

$$\begin{aligned} &\sum_{\xi \in F} H(E(A),-t)_{\mu,\xi} H(A,t)_{\xi,\nu} \\ &\quad= \sum_{\mu \leq_\Psi \xi \leq_\Psi \nu} (-1)^{d_\Psi(\mu,\xi)} t^{d_\Psi(\xi,\nu)+d_\Psi(\mu,\xi)} [P_\nu : S_\xi] \dim \operatorname{Ext}^{d_\Psi(\mu,\xi)}_{A-\mathrm{mod}_f}(S_\xi, S_\mu) \\ &\quad= t^{d_\Psi(\mu,\nu)} \sum_{j \geq 0} \sum_{\mu \leq_\Psi \xi \leq_\Psi \nu} (-1)^j [P_\nu : S_\xi] \dim \operatorname{Ext}^j_{A-\mathrm{mod}_f}(S_\xi, S_\mu) \\ &\quad= t^{d_\Psi(\mu,\nu)} \sum_{j \geq 0} (-1)^j \dim \operatorname{Ext}^j_{A-\mathrm{mod}_f}(P_\nu, S_\mu) \\ &\quad= \delta_{\mu,\nu} t^{d_\Psi(\mu,\nu)} = \delta_{\mu,\nu}. \end{aligned}$$

Thus, the matrix $H(E(A),-t)$ is the inverse of the matrix $H(A,t)$ and so $A$ is Koszul by [2, Theorem 2.11.1]. ∎

## 4. Koszulity of $\mathbf{S}_\Psi^{\mathfrak g}$

**4.1.** We shall use the following elementary result repeatedly.

**Lemma.** *Let $M \in \operatorname{Ob}\mathcal{F}(\mathfrak g)$. There exists an isomorphism*
$$M \cong \bigoplus_{\nu\in P^+} V(\nu)\otimes (V(\nu)^*\otimes M)^{\mathfrak g},$$
*of $\mathfrak g$-modules. In particular, if $M' \in \operatorname{Ob}\mathcal{F}(\mathfrak g)$, then we have*
$$(M'\otimes M)^{\mathfrak g} \cong \bigoplus_{\nu\in P^+} (M'\otimes V(\nu))^{\mathfrak g}\otimes (V(\nu)^*\otimes M)^{\mathfrak g}.$$

*Proof.* It suffices to prove the Lemma in the case when $M$ is a direct sum of copies of $V(\xi)$ for some $\xi \in P^+$. In this case, it is clear that the map $V(\xi)\otimes \operatorname{Hom}_{\mathfrak g}(V(\xi), M) \to M$ given by extending $v\otimes f \mapsto f(v)$, induces an isomorphism of $\mathfrak g$-modules. The Lemma follows by noticing that $\operatorname{Hom}_{\mathfrak g}(V(\xi), M) \cong (V(\xi)^*\otimes M)^{\mathfrak g}$. □

**4.2.** Consider the canonical surjection $\Pi_S : T(\mathfrak g) \to S(\mathfrak g)$ and let $\mathbf{\Pi}_S : \mathbf{T} \to \mathbf{S}$ be the map $1\otimes \Pi_S \otimes 1$. Identify the $\mathfrak g$-submodule of $T^2(\mathfrak g)$ spanned by the elements $\{x\otimes y - y\otimes x : x, y \in \mathfrak g\}$ (respectively, by the elements $\{x\otimes y + y\otimes x : x, y \in \mathfrak g\}$) with $\bigwedge^2\mathfrak g$ (respectively, with $S^2(\mathfrak g)$). Then $\ker \Pi_S$ is the ideal generated by $\bigwedge^2\mathfrak g$. Recall also that we have fixed $\psi \in P$ such that $\Psi = \Psi(\psi)$ is a subset of $R^+$. We say that a subset $F$ of $P^+$ is interval closed if for $\mu, \nu \in F$ with $\mu \le_\Psi \nu$ we have $[\mu, \nu]_\Psi \subset F$. It is clear that $\mathbf{\Pi}_S : \mathbf{T}_\Psi(F) \to \mathbf{S}_\Psi(F)$. We first prove

**Lemma.** (i) *The kernel of the restriction $\mathbf{\Pi}_S^{\mathfrak g} : \mathbf{T}^{\mathfrak g} \to \mathbf{S}^{\mathfrak g}$ is generated by $(\mathbf{V}^\circledast \otimes \bigwedge^2\mathfrak g\otimes \mathbf{V})^{\mathfrak g} \subset \mathbf{T}^{\mathfrak g}[2]$.*

(ii) *Let $F \subset P^+$ be interval closed. The kernel of $\mathbf{\Pi}_S^{\mathfrak g}$ restricted to $\mathbf{T}_\Psi^{\mathfrak g}(F)$ is generated by $(\mathbf{V}^\circledast \otimes \bigwedge^2\mathfrak g\otimes \mathbf{V})^{\mathfrak g} \cap \mathbf{T}_\Psi^{\mathfrak g}(F)[2]$.*

*Analogous statements hold for $\mathbf{E}^{\mathfrak g}$ and $\mathbf{E}_\Psi^{\mathfrak g}(F)$.*

*Proof.* Note that for $k = 0, 1$ we have
$$\mathbf{T}^{\mathfrak g}[k] = \mathbf{S}^{\mathfrak g}[k],$$
and hence $\ker \mathbf{\Pi}_S^{\mathfrak g}[k] = 0$ in these cases. For $k \ge 2$, we have
$$\ker \mathbf{\Pi}_S^{\mathfrak g} \cap \mathbf{T}^{\mathfrak g}[k] = (\mathbf{V}^\circledast \otimes \ker \Pi_S[k]\otimes \mathbf{V})^{\mathfrak g} = \sum_{j=0}^{k-2} (\mathbf{V}^\circledast \otimes (T^j(\mathfrak g)\otimes \textstyle\bigwedge^2\mathfrak g\otimes T^{k-j-2}(\mathfrak g))\otimes \mathbf{V})^{\mathfrak g}.$$
Part (i) of the Lemma follows if we prove for all $\lambda, \mu \in P^+$ and for all $j, k$ with $k \ge 2$ and $0 \le j \le k-2$, that
$$1_\lambda(\mathbf{V}^\circledast \otimes (T^j(\mathfrak g)\otimes \textstyle\bigwedge^2\mathfrak g\otimes T^{k-j-2}(\mathfrak g))\otimes \mathbf{V})^{\mathfrak g}1_\mu \subset 1_\lambda \mathbf{T}^{\mathfrak g}[j](\mathbf{V}^\circledast \otimes \bigwedge^2\mathfrak g\otimes \mathbf{V})^{\mathfrak g}\mathbf{T}^{\mathfrak g}[k-j-2]1_\mu,$$
or that
$$\begin{aligned}&(V(\lambda)^*\otimes (T^j(\mathfrak g)\otimes \textstyle\bigwedge^2\mathfrak g\otimes T^{k-j-2}(\mathfrak g))\otimes V(\mu))^{\mathfrak g}\\ &= \sum_{\nu,\xi\in P^+} (V(\lambda)^*\otimes T^j(\mathfrak g)\otimes V(\nu))^{\mathfrak g}(V(\nu)^*\otimes \textstyle\bigwedge^2\mathfrak g\otimes V(\xi))^{\mathfrak g}(V(\xi)^*\otimes T^{k-j-2}(\mathfrak g)\otimes V(\mu))^{\mathfrak g}\end{aligned} \tag{4.1}$$

(the product in the right hand side is taken in $\mathbf{T}^{\mathfrak{g}}$). But this follows from Lemma 4.1 and the proof of part (i) is complete.

To prove (ii) suppose now that $\lambda \le_\Psi \mu$ and that $k = d_\Psi(\lambda,\mu)$. Again if $k = 0,1$ there is nothing to prove. If $k \ge 2$, then by Lemma 2.1 we see that

$$(V(\lambda)^* \otimes T^j(\mathfrak{g}) \otimes V(\nu))^{\mathfrak{g}} \neq 0 \implies \nu - \lambda = \sum_{\alpha\in R} m_\alpha \alpha, \qquad m_\alpha \in \mathbf{Z}_+,\ \sum_{\alpha\in R} m_\alpha \le j,$$
$$(V(\nu)^* \otimes {\textstyle\bigwedge}^2\mathfrak{g} \otimes V(\xi))^{\mathfrak{g}} \neq 0 \implies \xi - \nu = \sum_{\alpha\in R} n_\alpha \alpha, \qquad n_\alpha \in \mathbf{Z}_+,\ \sum_{\alpha\in R} n_\alpha \le 2,$$
$$(V(\xi)^* \otimes T^{k-j-2}(\mathfrak{g}) \otimes V(\mu))^{\mathfrak{g}} \neq 0 \implies \mu - \xi = \sum_{\alpha\in R} \ell_\alpha \alpha, \qquad \ell_\alpha \in \mathbf{Z}_+,\ \sum_{\alpha\in R} \ell_\alpha \le k-j-2.$$

But now Lemma 2.3 gives

$$\alpha \notin \Psi \implies m_\alpha = n_\alpha = \ell_\alpha = 0, \qquad \sum_{\alpha\in\Psi} m_\alpha + n_\alpha + \ell_\alpha = j.$$

This means that

$$\lambda \le_\Psi \nu \le_\Psi \xi \le_\Psi \mu, \qquad d_\Psi(\lambda,\nu) = j, \qquad d_\Psi(\nu,\xi) = 2, \qquad d_\Psi(\xi,\mu) = k-j-2,$$

and since $F$ is interval closed we get $\nu, \xi \in F$. In other words, we have proved that in the case when $\lambda \le_\Psi \mu$ and $d_\Psi(\lambda,\mu) = k$, the right hand side of (4.1) is in the ideal of $\mathbf{T}^{\mathfrak{g}}_\Psi(F)$ generated by $(\mathbf{V}^{\circledast} \otimes \bigwedge^2\mathfrak{g} \otimes \mathbf{V}) \cap \mathbf{T}^{\mathfrak{g}}_\Psi(F)[2]$, which establishes part (ii) of the Lemma. $\square$

**4.3.** Before we continue proving the main result of this section, we summarize for the reader's convenience some standard facts about $\mathbf{Z}_+$-graded associative algebras. The details can be found in [2].

Suppose that $A = \bigoplus_{r\in\mathbf{Z}_+} A[r]$ be a $\mathbf{Z}_+$-graded associative algebra and that $A[0]$ is semi-simple. Clearly $A[r]$ is an $A[0]$-bimodule for all $r \ge 0$ and we let $T^r_{A[0]}(A[1])$ be the $r$-fold tensor product of the $A[0]$-bimodule $A[1]$ over $A[0]$. Setting,

$$T^0_{A[0]}(A[1]) = A[0], \qquad T_{A[0]}(A[1]) = \bigoplus_{r\in\mathbf{Z}_+} T^r_{A[0]}(A[1]),$$

we see that $T_{A[0]}(A[1])$ is a $\mathbf{Z}_+$-graded associative algebra and the assignment $\mathbf{m}(a) = a$, $a \in A[r]$, $r = 0,1$ extends to a canonical homomorphism $\mathbf{m} : T_{A[0]}(A[1]) \to A$ of $\mathbf{Z}_+$-graded associative algebras and $A[0]$-bimodules. The algebra $A$ is said to be *quadratic* if $\mathbf{m}$ is surjective and $\ker\mathbf{m}$ is generated by $\ker\mathbf{m} \cap T^2_{A[0]}(A[1])$.

The Koszul complex for a quadratic algebra $A$ is constructed as follows. Set

$$N^0_A = A[0], \qquad N^1_A = A[1],$$
$$N^r_A = \bigcap_{j=0}^{r-2} T^j_{A[0]}(A[1]) \otimes_{A[0]} (\ker\mathbf{m} \cap T^2_{A[0]}(A[1])) \otimes_{A[0]} T^{r-j-2}_{A[0]}(A[1]), \qquad r \ge 2.$$

Regard $N^r$ as a graded left $A$-module concentrated in degree $r$ via the canonical projection $A \twoheadrightarrow A/\bigoplus_{j>0} A[j] \cong A[0]$. Define a map

$$\partial_r : A \otimes_{A[0]} N^r_A \to A \otimes_{A[0]} N^{r-1}_A$$

by restricting the map

$$A \otimes_{A[0]} T^r_{A[0]}(A[1]) \to A \otimes_{A[0]} T^{r-1}_{A[0]}(A[1])$$
$$x \otimes a_1 \otimes \cdots \otimes a_r \mapsto x a_1 \otimes a_2 \cdots \otimes a_r, \qquad x \in A,\ a_j \in A[1],\ 1 \le j \le r.$$

Then $\partial_r$ is a homomorphism of graded left $A$-modules and of $A[0]$-bimodules and $\partial_{r-1}\partial_r = 0$. The complex of $\mathbf{Z}_+$-graded left $A$-modules

$$\cdots \xrightarrow{\partial_{r+1}} A \otimes_{A[0]} N^r_A \xrightarrow{\partial_r} A \otimes_{A[0]} N^{r-1}_A \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_2} A \otimes_{A[0]} A[1] \xrightarrow{\partial_1} A$$

called the Koszul complex of $A$. By [2, Theorem 2.6.1], a quadratic algebra is Koszul if and only if its Koszul complex is exact.

**4.4.** Let $F$ be a subset of $P^+$ and assume that $F$ is interval closed. Recall that

$$\mathbf{T}^{\mathfrak g}[0] = \bigoplus_{\lambda \in P^+} \mathbf{C}1_\lambda, \quad \mathbf{T}^{\mathfrak g}_\Psi(F)[0] = \bigoplus_{\lambda \in F} \mathbf{C}1_\lambda,$$

is a semisimple, commutative algebra. Clearly, if $M, N$ are $\mathbf{T}^{\mathfrak g}[0]$-bimodules, then for any $\lambda \in P^+$ we have,

$$M1_\lambda \otimes_{\mathbf{T}^{\mathfrak g}[0]} 1_\lambda N = M1_\lambda \otimes 1_\lambda N.$$

Further, given a $\mathbf{T}^{\mathfrak g}_\Psi(F)[0]$-bimodule $M$, we can always regard $M$ as a $\mathbf{T}^{\mathfrak g}[0]$-bimodule, by letting $1_\mu$, $\mu \notin F$ act trivially and in that case we have

$$M \otimes_{\mathbf{T}^{\mathfrak g}[0]} N = M \otimes_{\mathbf{T}^{\mathfrak g}_\Psi(F)[0]} N.$$

In particular, this implies that $T_{\mathbf{T}^{\mathfrak g}_\Psi(F)[0]}(\mathbf{T}^{\mathfrak g}_\Psi(F)[1])$ is canonically isomorphic to a subalgebra of $T_{\mathbf{T}^{\mathfrak g}[0]}(\mathbf{T}^{\mathfrak g}[1])$ and the restriction of $\mathbf{m}$ maps it to $\mathbf{T}^{\mathfrak g}_\Psi(F)$.

Furthermore, we get

$$\mathbf{T}^{\mathfrak g}[1] \otimes_{\mathbf{T}^{\mathfrak g}[0]} \mathbf{T}^{\mathfrak g}[1] \cong \bigoplus_{\lambda,\mu,\nu \in P^+} (V(\lambda)^* \otimes \mathfrak g \otimes V(\mu))^{\mathfrak g} \otimes (V(\mu)^* \otimes \mathfrak g \otimes V(\nu))^{\mathfrak g},$$

$$\mathbf{T}^{\mathfrak g}_\Psi(F)[1] \otimes_{\mathbf{T}^{\mathfrak g}[0]} \mathbf{T}^{\mathfrak g}_\Psi(F)[1] \cong \bigoplus_{(\lambda,\mu,\nu) \in \mathbf{F}} (V(\lambda)^* \otimes \mathfrak g \otimes V(\mu))^{\mathfrak g} \otimes (V(\mu)^* \otimes \mathfrak g \otimes V(\nu))^{\mathfrak g},$$

where

$$\mathbf{F} = \{(\lambda,\mu,\nu) \in F^3 : \lambda \le_\Psi \mu \le_\Psi \nu,\ d_\Psi(\lambda,\mu) = d_\Psi(\mu,\nu) = 1\}.$$

Applying Lemma 4.1 now gives,

$$\mathbf{T}^{\mathfrak g}[1] \otimes_{\mathbf{T}^{\mathfrak g}[0]} \mathbf{T}^{\mathfrak g}[1] \cong \bigoplus_{\lambda,\nu \in P^+} (V(\lambda)^* \otimes \mathfrak g \otimes \mathfrak g \otimes V(\nu))^{\mathfrak g},$$
$$\cong \mathbf{T}^{\mathfrak g}[2],$$

while an argument identical to the one given in the proof of Lemma 4.2(ii) shows that

$$\mathbf{T}^{\mathfrak g}_\Psi(F)[1] \otimes_{\mathbf{T}^{\mathfrak g}_\Psi[0]} \mathbf{T}^{\mathfrak g}_\Psi(F)[1] \cong \mathbf{T}^{\mathfrak g}_\Psi(F)[2].$$

More generally, we see now that we have an isomorphism of $\mathbf{T}^{\mathfrak g}[0]$-bimodules,

$$(\mathbf{T}^{\mathfrak g}[1])^{\otimes k}_{\mathbf{T}^{\mathfrak g}[0]} \cong \mathbf{T}^{\mathfrak g}[k], \qquad (\mathbf{T}^{\mathfrak g}_\Psi(F)[1])^{\otimes k}_{\mathbf{T}^{\mathfrak g}[0]} \cong \mathbf{T}^{\mathfrak g}_\Psi(F)[k], \qquad k \ge 1.$$

The first statement of the following proposition is now immediate while the second follows by using Lemma 4.2.

**Proposition.** *The map* $\mathbf{m} : T_{\mathbf{T}^{\mathfrak{g}}[0]}(\mathbf{T}^{\mathfrak{g}}[1]) \to \mathbf{T}^{\mathfrak{g}}$ *is an isomorphism of* $\mathbf{Z}_+$*-graded associative algebras. In particular, the algebras* $\mathbf{S}^{\mathfrak{g}}$, $\mathbf{E}^{\mathfrak{g}}$ *are quadratic. Similarly, for* $F \subset P^+$ *interval closed the restriction* $\mathbf{m} : T_{\mathbf{T}^{\mathfrak{g}}_{\Psi}(F)[0]}(\mathbf{T}^{\mathfrak{g}}_{\Psi}(F)[1]) \to \mathbf{T}^{\mathfrak{g}}_{\Psi}(F)$ *is an isomorphism of* $\mathbf{Z}_+$*-graded associative algebras. In particular, the algebras* $\mathbf{S}^{\mathfrak{g}}_{\Psi}(F)$, $\mathbf{E}^{\mathfrak{g}}_{\Psi}(F)$ *are quadratic.* □

**4.5.**

**Proposition.** *Let* $\lambda \in P^+$ *and assume that* $\Psi = \Psi(\psi)$ *for some* $\psi \in P$. *The algebras* $\mathbf{S}^{\mathfrak{g}}_{\Psi}$ *and* $\mathbf{S}^{\mathfrak{g}}_{\Psi}(\lambda \leq_{\Psi})$ *are Koszul.*

*Proof.* We use the notation of Section 4.3. Let $A = \mathbf{S}^{\mathfrak{g}}_{\Psi}$ and for $F \subset P^+$ set $A(F) = \mathbf{S}^{\mathfrak{g}}_{\Psi}(F)$. Clearly $A = A(P^+)$. In Proposition 3.8 we proved that the algebra $A(\leq_{\Psi} \mu)$ is Koszul for all $\mu \in P^+$. Hence the Koszul complex of this algebra is exact. We now describe the relationship between the Koszul complex of $A$ and that of $A(\leq_{\Psi} \mu)$, $\mu \in P^+$.

Setting $N^r = N^r_A$ and $N^r(F) = N^r_{A(F)}$, we see that

$$N^r 1_\mu = \bigoplus_{\nu \leq_{\Psi} \mu} 1_\nu N^r 1_\mu \;\cong\; N^r(\leq_{\Psi} \mu),$$

$$N^r = \bigoplus_{\mu \in P^+} N^r 1_\mu \;\cong\; \bigoplus_{\mu \in P^+} N^r(\leq_{\Psi} \mu).$$

This gives,

$$A \otimes_{A[0]} N^r 1_\mu = \bigoplus_{\nu \leq_{\Psi} \mu} A 1_\nu \otimes_{A[0]} 1_\nu N^r 1_\mu = \bigoplus_{\nu, \xi \leq_{\Psi} \mu} A 1_\nu \otimes_{A[0]} 1_\xi N^r 1_\mu,$$

which in turn implies that

$$A \otimes_{A[0]} N^r 1_\mu \cong A(\leq_{\Psi} \mu) \otimes_{A(\leq_{\Psi} \mu)[0]} N^r(\leq_{\Psi} \mu).$$

Moreover this isomorphism is compatible with the maps $\partial_r$ and hence we get

$$\ker \partial_r|_{A \otimes_{A[0]} N^r 1_\mu} \subset \partial_{r+1}(A \otimes_{A[0]} N^{r+1} 1_\mu),$$

which proves that the Koszul complex for $A$ is exact. The proof when $F$ is the set $\lambda \leq_{\Psi}$ is similar and we omit the details. □

## 5. Quadratic dual of the algebra $\mathbf{S}^{\mathfrak{g}}_{\Psi}$

There are two notions of quadratic duals which appear in the literature. One definition can be found in the study of Koszul quotients of path algebras of quivers. The other one given in [2, Definition 2.8.1] applies to quadratic algebras $A$ which satisfy the additional condition that $A[r]$, for all $r \geq 0$ is a finitely generated left $A[0]$-module. In our case the two definitions are equivalent for the algebras $\mathbf{S}^{\mathfrak{g}}_{\Psi}(F)$ when $F$ is finite and interval closed but only the first definition can be used to define the quadratic dual of $\mathbf{S}^{\mathfrak{g}}_{\Psi}$.

**5.1.** Fix $\Psi = \Psi(\psi)$, $\psi \in P$.

**Lemma.** *Let $F \subset P^+$ be interval closed. Let $\mathbf{A}$ be $\mathbf{T}$, $\mathbf{S}$ or $\mathbf{E}$. Then $(\mathbf{A}^{\mathfrak{g}}_{\Psi}(F))^{op}$ is isomorphic to the the subalgebra*

$$\bigoplus_{\lambda \le_\Psi \mu \in F} 1_\mu \mathbf{A}^{\mathfrak{g}}[d_\Psi(\lambda,\mu)]1_\lambda \subset \mathbf{A}^{\mathfrak{g}},$$

*of $\mathbf{A}_\Psi$.*

*Proof.* Note that $A[k]^* \cong A[k]$ as $\mathfrak{g}$-modules if $A$ is one of $T(\mathfrak{g})$, $S(\mathfrak{g})$ or $\bigwedge \mathfrak{g}$. Moreover for $M, N \in \operatorname{Ob} \mathcal{F}(\mathfrak{g})$, we have $(M^* \otimes N)^{\mathfrak{g}} \cong (N^* \otimes M)^{\mathfrak{g}}$. Furthermore, if $K \in \operatorname{Ob}\mathcal{F}(\mathfrak{g})$, then the canonical map (cf. Lemma 2.1)

$$(M^* \otimes N)^{\mathfrak{g}} \otimes (N^* \otimes K)^{\mathfrak{g}} \to (M^* \otimes K)^{\mathfrak{g}}$$

induces the canonical map

$$(K^* \otimes N)^{\mathfrak{g}} \otimes (N^* \otimes M)^{\mathfrak{g}} \to (K^* \otimes M)^{\mathfrak{g}}.$$

Let $\lambda \le_\Psi \mu \in F$. By the above we have an isomorphism of vector spaces

$$\begin{aligned} 1_\lambda \mathbf{A}^{\mathfrak{g}}[d_\Psi(\lambda,\mu)]1_\mu &= (V(\lambda)^* \otimes A[d_\Psi(\lambda,\mu)] \otimes V(\mu))^{\mathfrak{g}} \\ &\cong (V(\mu)^* \otimes A[d_\Psi(\lambda,\mu)] \otimes V(\lambda))^{\mathfrak{g}} = 1_\mu \mathbf{A}^{\mathfrak{g}}[d_\Psi(\lambda,\mu)]1_\lambda \end{aligned}$$

which extends to the isomorphism of algebras

$$\bigoplus_{\lambda \le_\Psi \mu \in F} 1_\mu \mathbf{A}^{\mathfrak{g}}[d_\Psi(\lambda,\mu)]1_\lambda \cong (\mathbf{A}^{\mathfrak{g}}_\Psi(F))^{op}.$$ □

**5.2.** Let $(\cdot,\cdot)_{\mathfrak{g}}$ be the Killing form of $\mathfrak{g}$. Define $(\cdot,\cdot)_{T(\mathfrak{g})} : T(\mathfrak{g}) \otimes T(\mathfrak{g}) \to \mathbf{C}$ by extending linearly the assignment

$$(T^r(\mathfrak{g}), T^s(\mathfrak{g})) = 0, \qquad r \ne s,$$
$$(x_1 \otimes \cdots \otimes x_r, y_r \otimes \cdots \otimes y_1)_{T(\mathfrak{g})} = (x_1,y_1)_{\mathfrak{g}} \cdots (x_r,y_r)_{\mathfrak{g}}, \qquad x_i, y_i \in \mathfrak{g},\ 1 \le i \le r.$$

Then $(\cdot,\cdot)_{T(\mathfrak{g})}$ is a $\mathfrak{g}$-invariant symmetric bilinear form on $T(\mathfrak{g})$. It is easy to check that the restrictions of $(\cdot,\cdot)_{T(\mathfrak{g})}$ to $S^2(\mathfrak{g}) \otimes S^2(\mathfrak{g})$ and $\bigwedge^2 \mathfrak{g} \otimes \bigwedge^2 \mathfrak{g}$ are non-degenerate. Since $T^2(\mathfrak{g}) = S^2(\mathfrak{g}) \oplus \bigwedge^2 \mathfrak{g}$ and

$$(x,y)_{T(\mathfrak{g})} = 0, \qquad x \in S^2(\mathfrak{g}),\ y \in \textstyle\bigwedge^2 \mathfrak{g},$$

it follows that

$$\{x \in T^2(\mathfrak{g}) \,:\, (x, \textstyle\bigwedge^2 \mathfrak{g})_{T(\mathfrak{g})} = 0\} = S^2(\mathfrak{g}). \tag{5.1}$$

**5.3.** Define

$$\langle \cdot,\cdot \rangle : \mathbf{T} \otimes \mathbf{T} \to \mathbf{C}$$

by

$$\langle f \otimes a \otimes v, f' \otimes b \otimes v' \rangle = (a,b)_{T(\mathfrak{g})} f'(v) f(v'), \qquad v, v' \in \mathbf{V},\ f, f' \in \mathbf{V}^{\circledast}, a, b \in T(\mathfrak{g}).$$

It is easy to check that $\langle\cdot,\cdot\rangle$ is a symmetric non-degenerate and $\mathfrak{g}$-invariant form. Moreover,

$$\langle 1_\lambda u a, v 1_\mu \rangle = \delta_{\lambda,\mu} \langle u, av \rangle, \qquad u, v \in \mathbf{T},\ a \in \mathbf{T}[0],\ \lambda, \mu \in P^+. \tag{5.2}$$

**Lemma.** *The restriction of $\langle\cdot,\cdot\rangle$ to $\mathbf{T}^{\mathfrak{g}} \otimes \mathbf{T}^{\mathfrak{g}}$ is non-degenerate. In particular, the restriction of $\langle\cdot,\cdot\rangle$ to $\mathbf{T}^{\mathfrak{g}}_\Psi(F) \otimes (\mathbf{T}^{\mathfrak{g}}_\Psi(F))^{op}$ is non-degenerate.*

*Proof.* Since

$$\mathbf{T}^{\mathfrak{g}} = \bigoplus_{\lambda,\mu\in P^+,k\in\mathbf{Z}_+} 1_\lambda \mathbf{T}^{\mathfrak{g}}[k] 1_\mu$$

and $\langle \mathbf{T}[k], \mathbf{T}[s]\rangle = 0$, $k \neq s$, it suffices to show that for all $\lambda, \mu \in P^+$,

$$x \in 1_\mu \mathbf{T}^{\mathfrak{g}}[k]1_\lambda, \qquad \langle x, 1_\mu \mathbf{T}^{\mathfrak{g}}[k]1_\lambda\rangle = 0 \implies \langle x, \mathbf{T}[k]\rangle = 0. \tag{5.3}$$

Let $y \in 1_\nu \mathbf{T}[k]1_\xi$. If $\nu \neq \mu$ or $\xi \neq \lambda$ then (5.3) follows from (5.2). Otherwise, write $y = \sum_{\zeta\in P^+} y_\zeta$, where $y_\zeta$ is in the $\zeta$-isotypical component of $1_\mu \mathbf{T}[k]1_\lambda \cong V(\mu)^* \otimes T(\mathfrak{g}) \otimes V(\lambda)$. Since the form is $\mathfrak{g}$-invariant, it follows now that $\langle x, y_\zeta\rangle = 0$ if $\zeta \neq 0$. This proves (5.3). The second assertion is immediate by Lemma 5.1. □

**5.4.** Let $F \subset P^+$ be interval closed. For $\mathbf{A} = \mathbf{S}$ or $\mathbf{E}$, let

$$\mathbf{R}_{\mathbf{A}} = \ker(\mathbf{T}^{\mathfrak{g}}_\Psi(F) \to \mathbf{A}^{\mathfrak{g}}_\Psi(F)) \cap \mathbf{T}^{\mathfrak{g}}[2], \quad \mathbf{R}^!_{\mathbf{A}} = \{x \in \mathbf{T}^{\mathfrak{g}}[2] \,:\, \langle \mathbf{R}_{\mathbf{A}}, x\rangle = 0\},$$

and set

$$(\mathbf{A}^{\mathfrak{g}}_\Psi(F))^! = (\mathbf{T}^{\mathfrak{g}}_\Psi(F))^{op}/\langle \mathbf{R}^!_{\mathbf{A}}\rangle.$$

If $F$ is finite, it is not hard to see that this algebra is isomorphic to the right quadratic dual of $\mathbf{A}^{\mathfrak{g}}_\Psi(F)$ as defined in [2].

**Proposition.** *Let $F \subset P^+$ be interval closed. Then $(\mathbf{S}^{\mathfrak{g}}_\Psi(F))^! \cong (\mathbf{E}^{\mathfrak{g}}_\Psi(F))^{op}$. In particular, for all $\lambda' \leq_\Psi \lambda \in P^+$, the algebras $\mathbf{E}^{\mathfrak{g}}_\Psi(\leq_\Psi \lambda)$, $\mathbf{E}^{\mathfrak{g}}_\Psi([\lambda', \lambda]_\Psi)$ are Koszul.*

*Proof.* To prove the first assertion, recall that by Lemma 4.2(ii)

$$\mathbf{R}_{\mathbf{S}} = \bigoplus_{\lambda\leq_\Psi\mu\in F\,:\,d_\Psi(\lambda,\mu)=2} (V(\lambda)^* \otimes {\textstyle\bigwedge}^2\mathfrak{g} \otimes V(\mu))^{\mathfrak{g}}.$$

Using Lemma 5.3 and (5.1), we conclude that

$$\mathbf{R}^!_{\mathbf{S}} = \bigoplus_{\lambda\leq_\Psi\mu\in F\,:\,d_\Psi(\lambda,\mu)=2} (V(\mu)^* \otimes S^2(\mathfrak{g}) \otimes V(\lambda))^{\mathfrak{g}}$$

It remains to apply Lemma 4.2(ii) and Lemma 5.1. The second assertion follows immediately from [2, Proposition 2.2.1]. □

**5.5.** The following proposition completes the proof of Theorem 1.

**Proposition.** *Let $\lambda \in P^+$. The algebras $\mathbf{E}^{\mathfrak{g}}_\Psi$, $\mathbf{E}^{\mathfrak{g}}_\Psi(\lambda \leq_\Psi)$ are Koszul.*

*Proof.* As in Proposition 4.5, it suffices to prove that the Koszul complex for $\mathbf{E}^{\mathfrak{g}}_\Psi$ is exact. The argument is similar to that in Proposition 4.5, with the algebras $\mathbf{S}$ replaced by the corresponding algebras $\mathbf{E}$ and is omitted. □

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